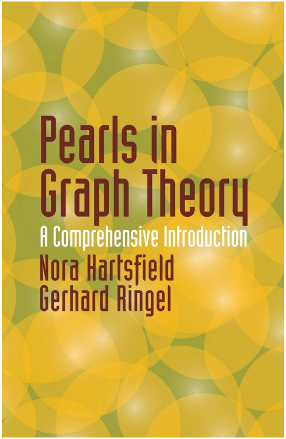
Introduction to Graph Theory

Chapter 10. Graphs on Surfaces

10.2. Planar Graphs Revisited—Proofs of Theorems



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Theorem 10.2.1

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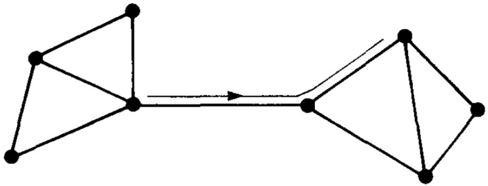

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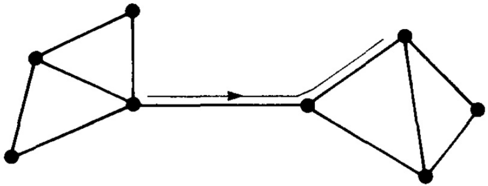

□

Theorem 10.2.2

Theorem 10.2.2. If H is a subgraph of a planar graph G, then H is planar.

Proof. Let G be a planar graph with p vertices, q edges, an planar rotation ρ (so we have $p - q + r(\rho) = 2$). We consider connected graph G and graph $H = G - e$ for some edge of G. Since each subgraph of G results from successively deleting edges (and isolated vertices) from G, then if we show that $H = G - e$ is planar, this is sufficient (by induction). We consider three cases.

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Case 1. Suppose one of the endpoints of e, say x, has degree 1. Then $G - e$ equals $G - x$ plus an isolated vertex (namely, vertex x), and so e is a bridge of G. Now $G - x$ has $p - 1$ vertices and $q - 1$ edges. Let $\hat{\rho}$ be the rotation of $G - x$ that is the same as ρ everywhere but at the other end of e, say vertex y, where edge e is deleted from the rotation. Now the circuit in G that contains edge e, must contain it in both directions by Theorem 10.2.1.

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Proof (continued). So this circuit in G has a corresponding circuit in $H = G - x$. Any other circuit in G, is also a circuit in $H = G - x$, so that $r(\rho) = r(\hat{\rho})$ (in $G - x$ and G, respectively). So for graph $G - v$, $(p - 1) - (q - 1) + r(\hat{\rho}) = p - q + r(\rho) = 2$, so that $G - v$ is planar and, hence $G - e$ is planar.

Case 2. Suppose edge e is a bridge of G. Denote the number of edges, vertices, and circuits on one bank as p_1, q_1, and $r_1(\hat{\rho})$, and on the other bank as p_2, q_2, and $r_2(\hat{\rho})$ (where $\hat{\rho}$ is the rotation that is the same as ρ everywhere, except at the ends of edge e, where edge e is deleted from the rotation). Then we have $p_1 + p_2 = p$, $q_1 + q_2 = q - 1$, and $r_1(\hat{\rho}) + r_2(\hat{\rho}) = r(\rho) + 1$ (by Theorem 10.2.1, edge e is in one circuit of G in both directions, so this circuit breaks into two circuits in $G - e$ with one circuit in each component of $G - e$).

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Theorem 10.2.2 (continued 2)

Theorem 10.2.2. If H is a subgraph of a planar graph G, then H is planar.

Proof (continued). Since G is planar then $p - q + r(\rho) = 2$, so that $p - (q - 1) + (r(\rho) + 1) = 4$, $(p_1 + p_2) - (q_1 + q_2) + (r_1(\hat{\rho}) + r_2(\hat{\rho})) = 4$, and so $(p_1 - q_1 + r_1(\hat{\rho})) + (p_2 - q_2 + r_2(\hat{\rho})) = 4$. By Theorem 10.1.2, $p_1 - q_1 + r_1(\hat{\rho}) \leq 2$ and $p_2 - q_2 + r_2(\hat{\rho}) \leq 2$, so we must have $p_1 - q_1 + r_1(\hat{\rho}) = 2$ and $p_2 - q_2 + r_2(\hat{\rho}) = 2$. So both banks of $G - e$ are planar, and hence $G - e$ is planar itself, as claimed.

Case 3. Suppose e is not a bridge. The graph $G - e$ has p vertices and $q - 1$ edges. Let $\hat{\rho}$ be the rotation of $G - e$ that is the same as ρ everywhere except at the ends of e, where edge e is deleted from the rotation. As in the proof of Theorem 10.1.2 and Figures 10.1.11 and 10.1.12 (below), we have that in $G - e$ either one circuit is replaced with two (Figure 10.1.11) or two circuits are replaced with one (Figure 10.1.12).

Theorem 10.2.2 (continued 2)

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Proof (continued). Since G is planar then $p - q + r(\rho) = 2$, so that $p - (q - 1) + (r(\rho) + 1) = 4$, $(p_1 + p_2) - (q_1 + q_2) + (r_1(\hat{\rho}) + r_2(\hat{\rho})) = 4$, and so $(p_1 - q_1 + r_1(\hat{\rho})) + (p_2 - q_2 + r_2(\hat{\rho})) = 4$. By Theorem 10.1.2, $p_1 - q_1 + r_1(\hat{\rho}) \leq 2$ and $p_2 - q_2 + r_2(\hat{\rho}) \leq 2$, so we must have $p_1 - q_1 + r_1(\hat{\rho}) = 2$ and $p_2 - q_2 + r_2(\hat{\rho}) = 2$. So both banks of $G - e$ are planar, and hence $G - e$ is planar itself, as claimed.

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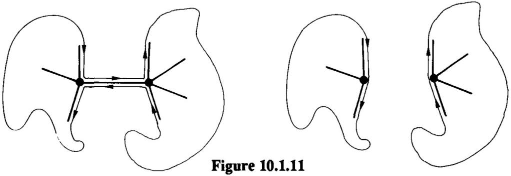

Figure 10.1.11

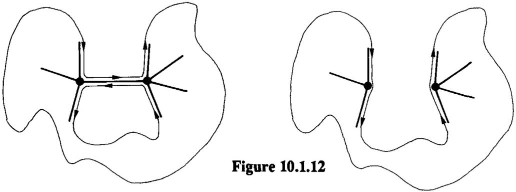

Figure 10.1.12

Theorem 10.2.2 (continued 3)

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Proof (continued). Hence, either $r(\hat{\rho}) = r(\rho) - 1$ or $r(\hat{\rho}) = r(\rho) + 1$. If $r(\hat{\rho}) = r(\rho) - 1$, then $p - (q - 1) + r(\hat{\rho}) = p - q + 1 + r(\rho) - 1 = 2$ and hence $G - e$ is planar. ASSUME $r(\hat{\rho}) = r(\rho) + 1$. Then $p - (q - 1) + r(\hat{\rho}) = p - q + 1 + r(\rho) + 1 = 4 > 2$, a CONTRADICTION to Theorem 10.1.2, since $G - e$ is connected. So we must have $r(\hat{\rho}) = r(\rho) - 1$, in which case $p - (q - 1) + r(\hat{\rho}) = p - q + 1 + r(\rho) - 1 = 2$ and hence $G - e$ is planar.

In all three cases, $G - e$ is planar and, as mentioned above, every subgraph of G is planar, as claimed. □

Theorem 10.2.3

Theorem 10.2.3. The complete bipartite graph $K_{3,3}$ is not planar.

Proof. ASSUME $K_{3,3}$ is planar. Then there is a rotation ρ of $K_{3,3}$ such that $p - q + r(\rho) = 2$. Since $p = 6$ and $q = 9$ for $K_{3,3}$, then we must have $r(\rho) = 5$. Since every edge is used twice in the 5 circuits induced by rotation ρ, then the average length of a circuit is $18/5 = 3\,3/5$. So at least one of the circuits has length 3.

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Theorem 10.2.5

Theorem 10.2.5. A maximal planar graph G with three or more vertices is connected and has no bridge.

Proof. Let G be a maximal planar graph with p vertices and q edges, and planar rotation ρ. ASSUME G is not connected. Then each of its components is planar since G is planar (by the definition of . Suppose one component has p_1 vertices, q_1 edges, and $r_1(\rho)$ circuits, and another component has p_2 vertices, q_2 edges, and $r_2(\rho)$ circuits. Then $p_1 - q_1 + r_1(\rho) = 2$ and $p_2 - q_2 + r_2(\rho) = 2$, by the definition of planar of a non-connected graph.

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Theorem 10.2.5 (continued 1)

Proof (continued). In the new component we have

$$(p_1 + p_2) - (q_1 + q_2 + 1) + (r_1(\rho) + r_2(\rho) - 1)$$

$$= (p_1 - q_1 + r_1(\rho)) + (p_2 - q_2 + r_2(\rho)) - 2 = 2 + 2 - 2 = 2,$$

so the new component is planar and hence the new graph with one more edge than graph G is planar. But this is a CONTRADICTION to the hypothesis that G is maximal planar. So the assumption that G is not connected is false, and hence G is connected, as claimed.

We now show that G cannot have a bridge, by contradiction. ASSUME G has a bridge. Since G has at least three vertices, one bank has more than one vertex. Denote the endpoints of the bridge by 1 and 2, and WLOG assume that vertex 3 is adjacent to vertex 1 and is on the same circuit as the bridge in a planar rotation of G (notice that the bridge is in only one circuit, by Theorem 10.2.1). See Figure 10.2.1 left (below).

Theorem 10.2.5 (continued 1)

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Theorem 10.2.5 (continued 2)

Proof (continued).

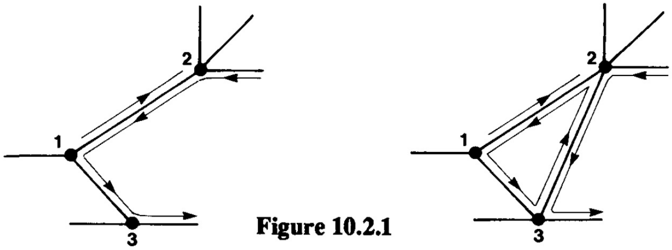

Figure 10.2.1

We now add an edge e joining vertices 2 and 3. Choose the rotation $\hat{\rho}$ of $G + e$ that is the same as ρ everywhere, except at vertices 2 and 3. If, in terms of a scheme representation of ρ at vertices 2 and 3, we have

$$2.\ \cdots x\ 1\ y \cdots \qquad \text{and} \qquad 3.\ \cdots z\ 1\ w \cdots,$$

then define $\hat{\rho}$ at vertices 2 and 3 as

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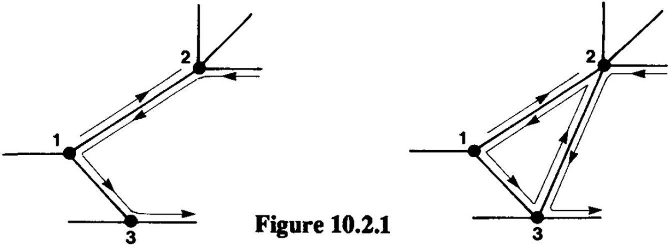

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Theorem 10.2.5 (continued 3)

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Proof (continued). We then have that $r(\hat{\rho}) = r(\rho) + 1$ (see Figure 10.2.1 right), since the one circuit containing the bridge in G becomes two circuits in $G - e$ (one of which is of length three). Graph $G + e$ has $q + 1$ edges, so $p - (q + 1) + r(\hat{\rho}) = p - q - 1 + r(\rho) + 1 = p - q + r(\rho) = 2$. But we have added an edge between two nonadjacent vertices in G, producing $G + e$ that is still planar. This CONTRADICTS the assumption that G is a maximal planar graph. So the assumption that G has a bridge is false, and hence G contains no bridge, as claimed.

We have shown that G is connected and G has no bridge, as claimed. □

Theorem 10.2.6

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

"Proof." We give a proof by contradiction. ASSUME G is is a maximal planar graph with planar rotation ρ and with the property that at least one circuit induced by ρ has length greater than three. Let G have p vertices and q edges. We consider three cases.

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Case 1. ASSUME there is a circuit of length five or more induced by rotation ρ with all vertices distinct (and, therefore, a cycle in G), as in Figure 10.2.2 (left) below. Then at least one pair of vertices on the cycle is not adjacent, since otherwise the given graph would contain a subgraph isomorphic to K_5, K_5 is not planar by Theorem 10.2.4, and then we would have that G is not planar by the contrapositive of Theorem 10.2.2.

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Theorem 10.2.6 (continued 1)

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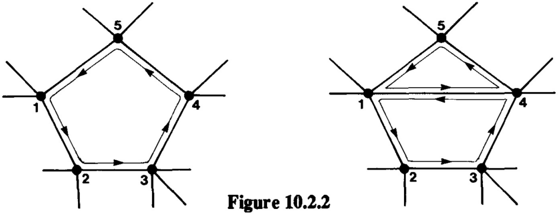
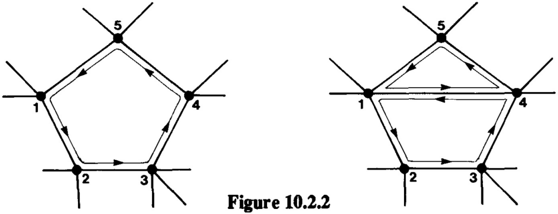


Figure 10.2.2

Suppose that vertex 1 is not adjacent to vertex 4. Add the edge e between vertices 1 and 4, obtaining a new graph $G + e$, and define the rotation $\hat{\rho}$ of $G + e$ to be the same as ρ everywhere, except at vertices 1 and 4. At vertices 1 and 4 the rotation is as suggested by Figure 10.2.2 right. Then $G + e$ has $q + 1$ edges and $r(\hat{\rho}) = r(\rho) + 1$, so that $p - (q + 1) + r(\hat{\rho}) = p - q - 1 + r(\rho) + 1 = p - q + r(\rho) = 2$, and $G + e$ is planar. But this is a CONTRADICTION, since G is a *maximal* planar graph. So the assumption that G has a circuit of length five or more with all vertices distinct is false.

Theorem 10.2.6 (continued 1)

"Proof" (continued).



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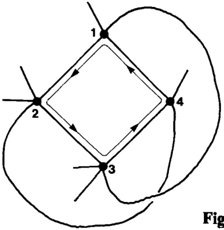
Theorem 10.2.6 (continued 2)

"Proof" (continued). Case 2. ASSUME there is a circuit induced by ρ of length four that is a cycle. If any two of the vertices of the cycle are not adjacent, then we add an edge as in Case 1 and get a contradiction. So suppose that all four vertices are mutually adjacent, as in Figure 10.2.3 left (notice that vertices 1 and 3 are adjacent, and vertices 2 and 4 are adjacent, though the drawing given here has a crossing).



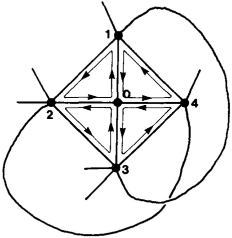


Figure 10.2.3

We construct a new graph $\hat{G}$ from G by adding a vertex 0 and connecting it to all four vertices of the cycle (Figure 10.2.3 right).

Theorem 10.2.6 (continued 3)

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

"Proof" (continued). Define the rotation $\hat{\rho}$ of $\hat{G}$ to be the same as ρ everywhere, except at vertices 1, 2, 3, 4, and 0. At these vertices, define $\hat{\rho}$ as suggested by Figure 10.2.3 right. So $\hat{G}$ has $p+1$ vertices, $q+4$ edges, and $r(\hat{\rho}) = r(\rho) + 3$. Thus, for $\hat{G}$ we have
$(p+1) - (q+4) + r(\hat{\rho}) = p - 3 - 3 + (r(\rho) + 3) = p - q + r(\rho) = 2$, since G is planar. Therefore $\hat{G}$ is planar. But $\hat{G}$ contains a subgraph isomorphic to K_5 on vertices 0, 1, 2, 3, 4, 5. But K_5 is not planar by Theorem 10.2.4, so that G is not planar by the contrapositive of Theorem 10.2.2. But this is a CONTRADICTION to the fact that G is planar. So the assumption that G contains a circuit induced by ρ of length four that is a cycle is false.

Theorem 10.2.6 (continued 4)

"Proof" (continued). Case 3. ASSUME G contains a circuit of length at least eight with a repeated edge, as in Figure 10.2.4 left.

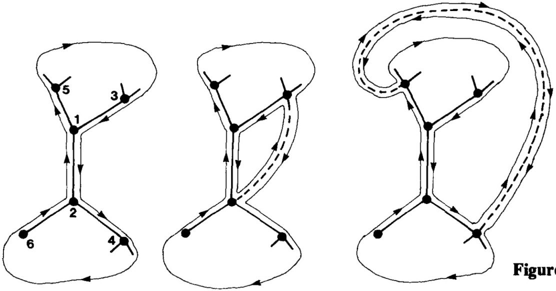


Figure 10.2.4

If any odd-numbered vertex is not adjacent to any even-numbered vertex, then add the edge joining these vertices and the new graph is still planar, giving a contradiction as in Case 1 (since this increases the number of circuits by one and increases the number of edges by one). So we must have all odd-numbered vertices adjacent to all even-numbered vertices.

Theorem 10.2.6 (continued 5)

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

"Proof" (continued). But then G contains a subgraph isomorphic to $K_{3,3}$. But $K_{3,3}$ is not planar by Theorem 10.2.3, so that G is not planar by the contrapositive of Theorem 10.2.2. But this is a CONTRADICTION to the fact that G is planar. So the assumption that G contains a circuit of length at least eight with a repeated edge is false.

We have shown that in G: there is no circuit induced by ρ of length four that is a cycle (in Case 2), there is no circuit of length five or more induced by rotation ρ with all vertices distinct (in Case 1), and there is no circuit of length at least eight with a repeated edge (in Case 3). From this, the text concludes that every circuit induced by ρ has length three. So. . . □?

Theorem 10.2.6 (continued 5)

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

“Proof” (continued). But then G contains a subgraph isomorphic to $K_{3,3}$. But $K_{3,3}$ is not planar by Theorem 10.2.3, so that G is not planar by the contrapositive of Theorem 10.2.2. But this is a CONTRADICTION to the fact that G is planar. So the assumption that G contains a circuit of length at least eight with a repeated edge is false.

We have shown that in G: there is no circuit induced by ρ of length four that is a cycle (in Case 2), there is no circuit of length five or more induced by rotation ρ with all vertices distinct (in Case 1), and there is no circuit of length at least eight with a repeated edge (in Case 3). From this, the text concludes that every circuit induced by ρ has length three. So. . . □?

Theorem 10.2.7

Theorem 10.2.7. In a maximal planar graph with p vertices, $p \geq 3$, and q edges we have $q = 3p - 6$.

Proof. By Theorem 10.2.6, for any planar rotation ρ of a maximal planar graph, every circuit induced by ρ has length three. Since every edge is present twice in the collection of circuits, and there are $r(\rho)$ circuits (each of length three), then $3r(\rho) = 2q$. Hence $p - q + r(\rho) = p - q + 2q/3 = 2$ (because the graph is planar), which implies $q = 3p - 6$, as claimed. □

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Theorem 10.2.9

Theorem 10.2.9. A connected graph that can be drawn in the plane with no crossings has a planar rotation. That is, graphs that are planar under the definition in Chapter 8 are also planar under the definition in Chapter 10.

Proof. We give a proof by picture. Let G be a connected graph with p vertices and q edges that can be drawn in the plane with no crossings. This results in some number of regions (or "faces") and we assign a rotation to G by giving every vertex a clockwise rotation. See Figure 10.2.5. Then the circuits induced by ρ are the boundaries of the faces in the drawing, and hence $r(\rho) = r$. □

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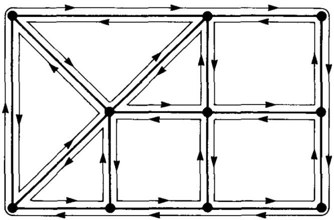

Figure 10.2.5

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Theorem 10.2.9. A connected graph that can be drawn in the plane with no crossings has a planar rotation. That is, graphs that are planar under the definition in Chapter 8 are also planar under the definition in Chapter 10.

Proof. We give a proof by picture. Let G be a connected graph with p vertices and q edges that can be drawn in the plane with no crossings. This results in some number of regions (or "faces") and we assign a rotation to G by giving every vertex a clockwise rotation. See Figure 10.2.5. Then the circuits induced by ρ are the boundaries of the faces in the drawing, and hence $r(\rho) = r$. □

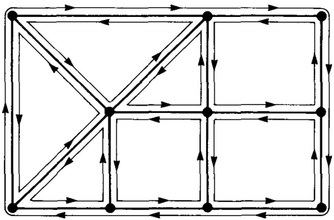

Figure 10.2.5