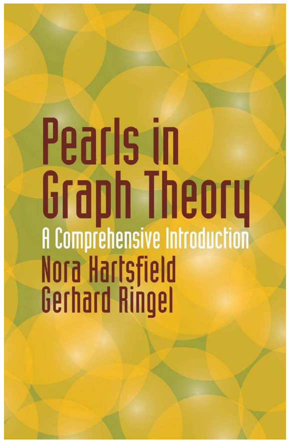
# Introduction to Graph Theory

**Chapter 10. Graphs on Surfaces**

10.3. The Genus of a Graph—Proofs of Theorems








## Theorem 10.3.3

**Theorem 10.3.3.** For the complete bipartite graph $K_{m,n}$,

$$\gamma(K_{m,n}) \geq \frac{(m-2)(n-2)}{4}.$$

**Proof.** Since $K_{m,n}$ is bipartite, then for any rotation $\rho$ of $K_{m,n}$, the shortest circuit induced by $\rho$ has length at least four (see Notes 10.3.C and 10.3.D). Since each edge appears twice in the circuits (once in each direction), then $2q \geq 4r(\rho)$ or $q/2 \geq r(\rho)$. In $K_{m,n}$ we have $p = m + n$ and $q = mn$, so that $q/2 = mn/2 \geq r(\rho)$. With $g = \gamma(K_{m,n})$ we have for a maximal rotation $\rho$ of $K_{m,n}$ that $p - q + r(\rho) = 2 - 2g$, or $(m+n) - (mn) + (mn/2) \geq p - q + r(\rho) = 2 - 2g$ or $2g \geq mn/2 - m - n + 2$ or $g \geq (mn - 2m - 2n + 4)/4$, so that $\gamma(K_{m,n}) \geq (m-2)(n-2)/4$, as claimed. □





## Corollary 10.3.A

**Corollary 10.3.A.** For the complete bipartite graph $K_{m,n}$ where $m$ and $n$ are both even,

$$\gamma(K_{m,n}) = \frac{(m-2)(n-2)}{4}.$$

**Proof.** Let the vertex set be $\{0, 2, 4, \ldots, m-2\} \cup \{1, 3, 5, \ldots, n-1\}$ where the partite sets consider of the even labeled vertices and the odd labeled vertices, respectively. Consider the rotation $\rho$ with the scheme:

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 (mod 4). | $1$ | $3$ | $5$ | $\cdots$ | $n-3$ | $n-1$ |
| 2 (mod 4). | $n-1$ | $n-3$ | $n-5$ | $\cdots$ | $3$ | $1$ |
| 1 (mod 4). | $0$ | $2$ | $4$ | $\cdots$ | $m-4$ | $m-2$ |
| 3 (mod 4). | $m-2$ | $m-4$ | $m-6$ | $\cdots$ | $2$ | $0$ |

It is to be shown in Exercise 10.3.5 that every induced cycle is of length four. So by Theorem 10.3.2, $\rho$ is a maximal rotation of $K_{m,n}$.





## Corollary 10.3.A (continued)

**Corollary 10.3.A.** For the complete bipartite graph $K_{m,n}$ where $m$ and $n$ are both even,

$$\gamma(K_{m,n}) = \frac{(m-2)(n-2)}{4}.$$

**Proof (continued).** With $g = \gamma(K_{m,n})$ we have for a maximal rotation $\rho$ that $p - q + r(\rho) = 2 - 2g$, or $(m+n) - (mn) + (mn/2) = p - q + r(\rho) = 2 - 2g$ or $2g = mn/2 - m - n + 2$ or $g = (mn - 2m - 2n + 4)/4$, so that $\gamma(K_{m,n}) = (m-2)(n-2)/4$, as claimed. □

## Theorem 10.3.5

**Theorem 10.3.5.** The genus of the complete graph satisfies the inequality

$$\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}.$$

**Proof.** Let $G$ be a graph with $p$ vertices, $q$ edges, and and maximal rotation $\rho$. Then be the new definition of genus $g$, we have $p - q + r(\rho) = 2 - 2g$. The shortest circuit possible is of length three (by Note 10.3.C), and every edge is used twice in circuits (once in each direction), so $2q \geq r(\rho)$. Therefore, $p - q + 2q/3 \geq 2 - 2g$, or $2g \geq 2 - p + q/3$. For $G = K_n$, then $p = n$ and $q = n(n-1)/2$, so that $2g \geq 2 - n + n(n-1)/6$ or $2g \geq 2 - n + n(n-1)/6 = (n^2 - 7n + 12)/6$, and hence $g \geq (n-3)(n-4)/12$ as claimed. □





## Theorem 10.3.7. Heawood's Theorem

**Theorem 10.3.7.** If $G$ is critical and $\gamma(G) \leq g$, where $g \geq 1$, then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

**Proof.** Let $G$ be a critical graph with chromatic number $\chi$. By Theorem 2.1.4, we have $(\chi - 1)p \leq 2q$. Since for any rotation we have $\gamma(G) \leq g$, then there exists a maximal rotation $\rho$ of $G$ such that $p - q + r(\rho) \geq 2 - 2g$ or $q - r(\rho) \leq p - (2 - 2g)$ or $3q - 3r(\rho) \leq 3p - 3(2 - 2g)$. The minimum possible length of a circuit is 3, so $2q \geq 3r(\rho)$ and we now have $3q \leq 3r(\rho) + 3p - 3(2 - 2g) \leq 2q + 3p - 3(2 - 2g)$ or $q \leq 3p - 6 + 6g$ or $2q \leq 6p - 12 + 12g$. Combining this with Theorem 2.1.4, we have $(\chi - 1)p \leq 6p - 12 + 12g$ or $\chi - 1 \leq 6 + (12g - 12)/p$. Since $g \geq 1$ (so that $12g - 12 \geq 0$) by hypothesis and $p \geq \chi$ (since $\chi$ involves vertex colorings; do that $/p \leq 1/\chi$), then we have $\chi - 1 \leq 6 + (12g - 12)/\chi$ and $\chi^2 - \chi \leq 5\chi + 12g - 12$ or $\chi^2 - 7\chi - (12g - 12) \leq 0$.





## Theorem 10.3.7. Heawood's Theorem (continued)

**Theorem 10.3.7.** If $G$ is critical and $\gamma(G) \leq g$, where $g \geq 1$, then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

**Proof.** ... $\chi^2 - 7\chi - (12g - 12) \leq 0$. By the quadratic formula, we can factor the inequality as

$$\left(\chi - \frac{7 + \sqrt{1 + 48g}}{2}\right)\left(\chi - \frac{7 - \sqrt{1 - 48g}}{2}\right) \leq 0.$$

Since $g \geq 1$, then $\sqrt{1 + 48g} \geq 7$ and $-\dfrac{7 - \sqrt{1 + 48g}}{2} \geq 0$. Since $\chi \geq 1$, the second factor is always positive. Since the product is nonpositive, then the first factor is at most 0 and hence we have

$$\chi - \frac{7 + \sqrt{1 + 48g}}{2} \leq 0 \text{ or } \chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

Since $\chi(G)$ is a whole number, we can round up, as claimed. □