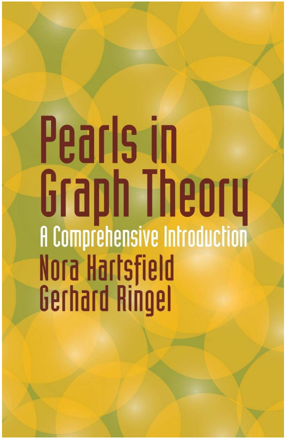
Introduction to Graph Theory

Chapter 10. Graphs on Surfaces

10.3. The Genus of a Graph—Proofs of Theorems



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$$\gamma(K_{m,n}) \geq \frac{(m-2)(n-2)}{4}.$$

Proof. Since $K_{m,n}$ is bipartite, then for any rotation ρ of $K_{m,n}$, the shortest circuit induced by ρ has length at least four (see Notes 10.3.C and 10.3.D). Since each edge appears twice in the circuits (once in each direction), then $2q \geq 4r(\rho)$ or $q/2 \geq r(\rho)$. In $K_{m,n}$ we have $p = m + n$ and $q = mn$, so that $q/2 = mn/2 \geq r(\rho)$.

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Corollary 10.3.A

Corollary 10.3.A. For the complete bipartite graph $K_{m,n}$ where m and n are both even,

$$\gamma(K_{m,n}) = \frac{(m-2)(n-2)}{4}.$$

Proof. Let the vertex set be $\{0, 2, 4, \ldots, m-2\} \cup \{1, 3, 5, \ldots, n-1\}$ where the partite sets consider of the even labeled vertices and the odd labeled vertices, respectively. Consider the rotation ρ with the scheme:

0 (mod 4).	1	3	5	$\cdots$	$n-3$	$n-1$
2 (mod 4).	$n-1$	$n-3$	$n-5$	$\cdots$	3	1
1 (mod 4).	0	2	4	$\cdots$	$m-4$	$m-2$
3 (mod 4).	$m-2$	$m-4$	$m-6$	$\cdots$	2	0

It is to be shown in Exercise 10.3.5 that every induced cycle is of length four. So by Theorem 10.3.2, ρ is a maximal rotation of $K_{m,n}$.

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Corollary 10.3.A (continued)

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Proof (continued). With $g = \gamma(K_{m,n})$ we have for a maximal rotation ρ that $p - q + r(\rho) = 2 - 2g$, or
$(m+n) - (mn) + (mn/2) = p - q + r(\rho) = 2 - 2g$ or
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Theorem 10.3.5

Theorem 10.3.5. The genus of the complete graph satisfies the inequality

$$\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}.$$

Proof. Let G be a graph with p vertices, q edges, and and maximal rotation ρ. Then be the new definition of genus g, we have $p - q + r(\rho) = 2 - 2g$. The shortest circuit possible is of length three (by Note 10.3.C), and every edge is used twice in circuits (once in each direction), so $2q \geq r(\rho)$. Therefore, $p - q + 2q/3 \geq 2 - 2g$, or $2g \geq 2 - p + q/3$. For $G = K_n$, then $p = n$ and $q = n(n-1)/2$, so that $2g \geq 2 - n + n(n-1)/6$ or $2g \geq 2 - n + n(n-1)/6 = (n^2 - 7n + 12)/6$, and hence $g \geq (n-3)(n-4)/12$ as claimed. □

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Theorem 10.3.7. Heawood's Theorem

Theorem 10.3.7. If G is critical and $\gamma(G) \leq g$, where $g \geq 1$, then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

Proof. Let G be a critical graph with chromatic number χ. By Theorem 2.1.4, we have $(\chi - 1)p \leq 2q$. Since for any rotation we have $\gamma(G) \leq g$, then there exists a maximal rotation ρ of G such that $p - q + r(\rho) \geq 2 - 2g$ or $q - r(\rho) \leq p - (2 - 2g)$ or $3q - 3r(\rho) \leq 3p - 3(2 - 2g)$. The minimum possible length of a circuit is 3, so $2q \geq 3r(\rho)$ and we now have $3q \leq 3r(\rho) + 3p - 3(2 - 2g) \leq 2q + 3p - 3(2 - 2g)$ or $q \leq 3p - 6 + 6g$ or $2q \leq 6p - 12 + 12g$.

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Theorem 10.3.7. Heawood's Theorem (continued)

Theorem 10.3.7. If G is critical and $\gamma(G) \leq g$, where $g \geq 1$, then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

Proof. $\ldots \chi^2 - 7\chi - (12g - 12) \leq 0$. By the quadratic formula, we can factor the inequality as

$$\left(\chi - \frac{7 + \sqrt{1 + 48g}}{2}\right)\left(\chi - \frac{7 - \sqrt{1 - 48g}}{2}\right) \leq 0.$$

Since $g \geq 1$, then $\sqrt{1 + 48g} \geq 7$ and $-\frac{7 - \sqrt{1 + 48g}}{2} \geq 0$. Since $\chi \geq 1$, the second factor is always positive. Since the product is nonpositive, then the first factor is at most 0 and hence we have

$$\chi - \frac{7 + \sqrt{1 + 48g}}{2} \leq 0 \text{ or } \chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

Since $\chi(G)$ is a whole number, we can round up, as claimed. □