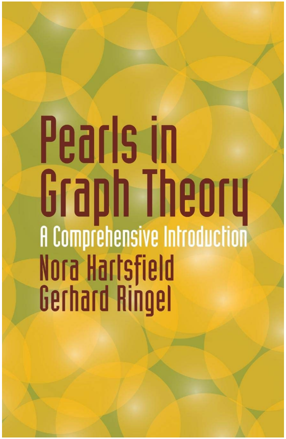
Introduction to Graph Theory

Chapter 2. Colorings of Graphs

2.1. Vertex Colorings—Proofs of Theorems



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Theorem 2.1.2

Theorem 2.1.2. Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

Proof. If G is critical, then we can take $H = G$. If G is not critical, then there is some proper subgraph H_1 of G with $\chi(H_1) = \chi(G)$ (by the definition of critical). If H_1 is not critical, then there exists a proper subgraph H_2 of H_1 such that $\chi(H_2) = \chi(H_1) = \chi(G)$ (again, by the definition of critical). Continuing this process of finding a chain of proper subgraphs (each with chromatic number equal to $\chi(G)$) there must be some $k \in \mathbb{N}$ such that subgraph H_k is critical, since G is finite. So $H = H_k$ is the desired critical subgraph. □

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Theorem 2.1.3

Theorem 2.1.3. If G is critical with chromatic number χ, then the degree of each vertex is at least $\chi - 1$.

Proof. Let G be a critical graph with chromatic number 4. ASSUME there is a vertex v of G where the degree of v is at most 2. Since G is critical and $G - e$ is a proper subgraph of G, then $G - v$ can be colored with only three colors. So color the vertices of $G - v$ with three colors, and color all vertices of G with the same colors, except for vertex v.

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Theorem 2.1.4

Theorem 2.1.4. If G is a critical graph with p vertices and q edges, and G has chromatic number χ, then the relation $(\chi - 1)p \leq 2q$ holds.

Proof. Let G be a critical graph. By Theorem 2.1.3, the degree of each vertex of G is at least $\chi - 1$. Since there are p vertices, then the sum of the degrees of the vertices of G is at least $(\chi - 1)p$. By Theorem 1.1.1, the sum of the degrees of the vertices of G is equal to $2q$. So $(\chi - 1)p \leq 2q$, as claimed. □

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Theorem 2.1.6

Theorem 2.1.6. A graph G is bipartite if and only if every cycle in G has even length.

Proof. First, suppose G is bipartite so that, by definition, $\chi(G) \leq 2$. ASSUME G contains an odd cycle C. Now $\chi(C) = 3$ and hence $\chi(G) \geq 3$, a CONTRADICTION. So G cannot contain an odd cycle. That is, every cycle in F has even length.

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Second, suppose G has no odd cycles. Without loss of generality we may assume G is connected (otherwise, we apply this argument to each component of G). Let x_0 be a vertex of G. We color G as follows. For vertex x of G, color x red if $d(x_0, x)$ is even and color x blue if $d(x_0, x)$ is odd. We must show that no two adjacent vertices have the same color.

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Theorem 2.1.6 (continued 1)

Proof (continued). Consider two adjacent vertices x and y.

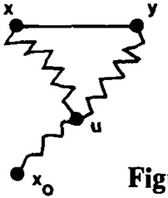

Figure 2.1.8

Choose a shortest path from x_0 to x and a shortest path from x_0 to y. Let u be the last common vertex in these shortest paths (see Figure 2.1.8). Vertex u may be equal to x_0, or u may also be x or y. Now we consider $d(u,x)$ and $d(u,y)$. If u is one of x or y, then either $d(u,x) = d(u,y) + 1$ (when $u = y$) or $d(u,x) = d(u,y) - 1$ (when $u = x$). In either case, one of the distances is odd and one is even (i.e., the distances have different parity). If u is not one of x or y, then the length of the cycle in Figure 2.1.8 is $d(u,x) + 1 + d(u,y)$.

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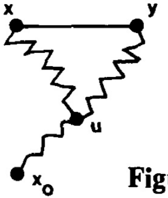

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Theorem 2.1.6 (continued 2)

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Proof (continued). In this case G has no odd cycles so this length, $d(u,x) + 1 + d(u,y)$, must be even. Hence $d(u,x)$ and $d(u,y)$ have different parity. Since the path from x_0 to x and the path from x_0 to y were chosen to be the shortest and since u lies on both paths, then

$$d(x_0, x) = d(x_0, u) + d(u, x) \text{ and } d(x_0) = d(x_0, u) + d(u, y).$$

So $d(x_0, x)$ and $d(x_0, y)$ also have different parity. Thus x and y receive different colors. Since x and y are arbitrary adjacent vertices of G, then the assignment of red and blue to the vertices of G is a coloring of G and hence $\chi(G) \leq 2$. That is, G is bipartite (by definition), as claimed. □

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