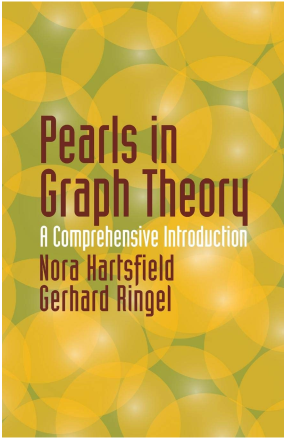
Introduction to Graph Theory

Chapter 2. Colorings of Graphs

2.2. Edge Colorings—Proofs of Theorems



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Theorem 2.2.1

Theorem 2.2.1. Let G be a graph. The number of colors required for a proper edge coloring of G is greater than or equal to the maximum degree of any vertex of G.

Proof. Let the maximum degree of a vertex in G be t. Then some vertex v of G is of degree t and so there are t edges of G incident to v. These t edges are therefore adjacent to each other and so must be of t different colors in a proper edge coloring of G. So the edge chromatic number of G is at least t, as claimed. □

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Theorem 2.2.3

Theorem 2.2.3. The edge chromatic number of K_{2n} is $2n - 1$.

Proof. Denote the vertices of K_{2n} as $0, 1, 2, \ldots, 2n-2, x$. Arrange the numbered vertices as a regular $(2n-1)$-gon with vertex x placed outside, as in Figure 2.2.3 (where $n = 5$).

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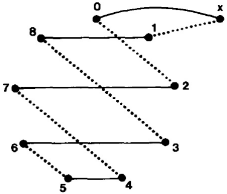

Figure 2.2.3

Let $C_1, C_2, \ldots, C_{2n-1}$ denote the $2n-1$ distinct colors. We color the following edges (represented as pairs of vertices)color C_1: $0\,x$, $1\,2n-2$, $2\,2n-3$, $\ldots$, $n\,n-1$; these edges are represented with solid segments in Figure 2.2.3.

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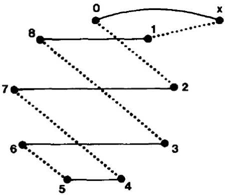

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Theorem 2.2.3 (continued)

Proof (continued). We determine the edges of color C_2 by leaving the outside vertex x fixed and "turning" the other ends of edges of color C_1 one unit clockwise (this produces the dotted segments in Figure 2.2.3). We continue this turning process to produce the following edges of the given color:

C_1	$0\,x$	$1\,2n-2$	$2\,2n-3$	$\cdots$	$n\,n-1$
C_2	$1\,x$	$2\,0$	$3\,2n-2$	$\cdots$	$n+1\,n$
C_3	$2\,x$	$3\,1$	$4\,0$	$\cdots$	$n+2\,n+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{2n-1}	$2n-2\,x$	$0\,2n-3$	$1\,2n-4$	$\cdots$	$n-1\,n-2$.

This gives a proper edge coloring of K_{2n} with the $2n-1$ colors $C_1, C_2, \ldots, C_{2n-1}$, so the edge chromatic number is at most $2n-1$. Since the maximum degree of a vertex in K_{2n} is $2n-1$, then by Theorem 2.2.1 we have that the edge chromatic number of K_{2n} equals $2n-1$, as claimed. □

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$$\begin{array}{cccccc} C_1 & 0\,x & 1\,2n-2 & 2\,2n-3 & \cdots & n\,n-1 \\ C_2 & 1\,x & 2\,0 & 3\,2n-2 & \cdots & n+1\,n \\ C_3 & 2\,x & 3\,1 & 4\,0 & \cdots & n+2\,n+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{2n-1} & 2n-2\,x & 0\,2n-3 & 1\,2n-4 & \cdots & n-1\,n-2. \end{array}$$

This gives a proper edge coloring of K_{2n} with the $2n-1$ colors $C_1, C_2, \ldots, C_{2n-1}$, so the edge chromatic number is at most $2n-1$. Since the maximum degree of a vertex in K_{2n} is $2n-1$, then by Theorem 2.2.1 we have that the edge chromatic number of K_{2n} equals $2n-1$, as claimed. □

Theorem 2.2.4

Theorem 2.2.4. The edge chromatic number of K_{2n-1} is $2n-1$.

Proof. Since by Theorem 2.2.3 we can properly edge color K_{2n} with $2n-1$ colors, we can properly edge color the subgraph K_{2n-1} of K_{2n} with $2n-1$ colors. We now show that K_{2n-1} cannot be properly edge colored using only $2n-2$ colors. Recall that there are $\binom{2n-1}{2} = (n-1)(2n-1)$ edges in K_{2n-1}. If these edges are colored by only $2n-2$ colors, then some color does to at least n edges (if not, then we only color $(n-1)(2n-2)$ edges and $(n-1)(2n-2) < (n-1)(2n-1)$).

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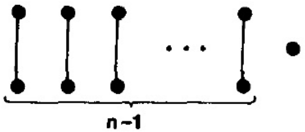

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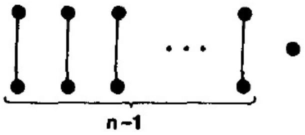

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Theorem 2.2.4 (continued)

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Proof (continued). So K_{2n-1} cannot be properly edge colored with only $2n-2$ colors. So by Vizing's Theorem (Theorem 2.2.2), since K_{2n-1} is $2n-2$-regular, the chromatic number of K_{2n-1} is $2n-1$, as claimed. □