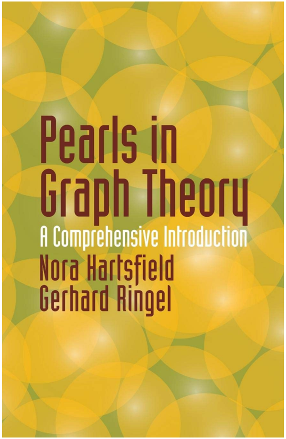
Introduction to Graph Theory

Chapter 2. Colorings of Graphs

2.3. Decompositions and Hamilton Cycles—Proofs of Theorems



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The complete graph K_{2n+1} has a decomposition into n Hamilton cycles.

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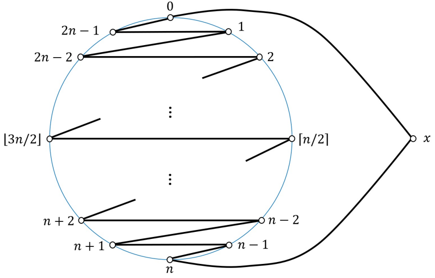

$$\begin{array}{cccccccccccc}
(C_1) & 0 & 2n-1 & 1 & 2n-2 & 2 & \cdots & n+1 & n-1 & n & x \\
(C_2) & 1 & 2n & 2 & 2n-1 & 3 & \cdots & n+2 & n & n+1 & x \\
\vdots & \vdots & & & & & & & & & \vdots \\
(C_n) & 2n-1 & 2n-2 & 0 & 2n-3 & 1 & \cdots & n & n-2 & n-1 & x.
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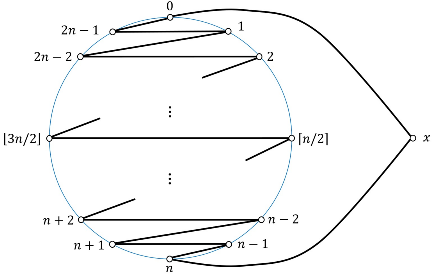

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□

Theorem 2.3.2

Theorem 2.3.2. K_{2n} has a decomposition into $n-1$ Hamilton cycles and a 1-factor.

Proof. In the proof of Theorem 2.2.3, we found the following color classes for colors $C_1, C_2, \ldots, C_{2n-1}$ in K_{2n}, where the vertices of K_{2n} are $x, 0, 1, 2, \ldots, 2n-2$:

$$\begin{array}{cccccc} C_1: & 0\,x & 1\,2n-2 & 2\,2n-3 & \cdots & n\,n-1 \\ C_2: & 1\,x & 2\,0 & 3\,2n-2 & \cdots & n+1\,n \\ C_3: & 2\,x & 3\,1 & 4\,0 & \cdots & n+2\,n+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{2n-1}: & 2n-2\,x & 0\,2n-3 & 1\,2n-4 & \cdots & n-1\,n-2. \end{array}$$

Combining the edges of color classes C_i and C_{i+1}, for $i = 1, 3, 5, \ldots, 2n-3$, we get the edges of $n-1$ Hamilton cycles in K_{2n}. The final color class C_{2n-1} then gives a 1-factor of K_{2n}, as claimed. □

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Proof. Let the vertices of K_{2n} be $0, 1, 2, \ldots, 2n-1$. Consider the graph K_{2n+1} with vertices $x, 0, 1, 2, \ldots, 2n-1$. A specific decomposition of K_{2n+1} into n Hamilton cycles is given in the proof of Theorem 2.3.1. By removing vertex x from K_{2n+1} and from each of the n Hamilton cycles, we get n Hamilton paths which give a decomposition of K_{2n}, as claimed. □

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Theorem 2.3.4

Theorem 2.3.4. The complete graph K_{2n} has a decomposition into $2n-1$ paths consisting of one path of each length k for $k = 1, 2, 3, \ldots, 2n-1$.

Proof. By Theorem 2.3.3, K_{2n} has a decomposition into n Hamilton paths. Each such path is of length $2n-1$ (there is a total of $n(2n-1) = 2n(2n-1)/2$ edges in the paths). Now a path of length $2n-1$ can be decomposed into a path of length i and a path of length $(2n-1)-i$ where $0 \leq i \leq n-1$. So for each i with $0 \leq i \leq n-1$ (a total of n values of i), decompose one of the Hamilton path into a path of length i and a path of length $(2n-1)-i$. This yields paths of length k for $k = 1, 2, 3, \ldots, 2n-1$, as claimed. □

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Theorem 2.3.5

Theorem 2.3.5. A snark has no Hamilton cycle.

Proof. Let G be a snark. Notice that since G is 3-regular and the sum of the degrees of the vertices of G is even (by Theorem 1.1.1), then the number of vertices must be even. ASSUME that G has a Hamilton cycle. Color the edges of the cycle with colors 1 and 2 alternating. Since there are an even number of vertices, this is a proper edge coloring of the cycle. Color the remaining edges in G with color 3.

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