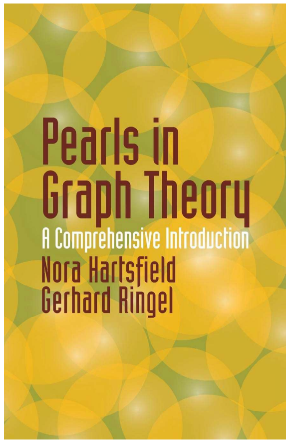
Introduction to Graph Theory

Chapter 2. Colorings of Graphs

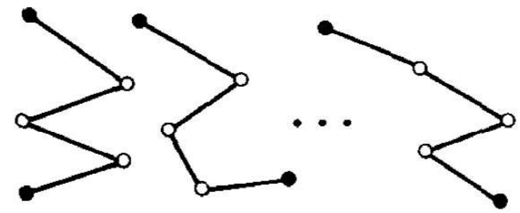
2.4. More Decompositions—Proofs of Theorems



Theorem 2.4.A

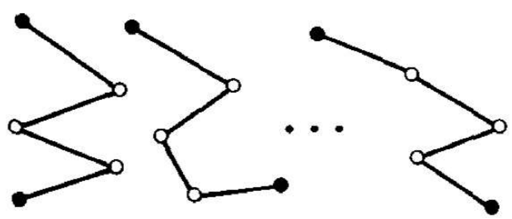
Theorem 2.4.A. Let G be a cubic graph. Then G has no decomposition into subgraphs, each of which is isomorphic to a path of length four.

Proof. Let G be a cubic graph. By Theorem 1.1.1 the sum of the degrees of the vertices is even so the number of vertices must be even, say there are $2m$ vertices. So the number of edges is $(2m)(3)/2 = 3m$. ASSUME G has a decomposition into paths of length 4. Then the number of edges $3m$ must be divisible by 4, so that $m = 4k$ for some $k \in \mathbb{N}$. Hence G has $2m = 8k$ vertices. The number of paths of length 4 in the decomposition must be $(3m)/4 = (12k)/4 = 3k$. Consider some of the paths of length 4:



Theorem 2.4.A (continued)

Proof (continued).



None of the white vertices (sometimes called *internal vertices* of the paths) can be identical, or we would have a vertex of degree 4 contradicting the fact that G is cubic. Now the total number of white (internal) vertices is

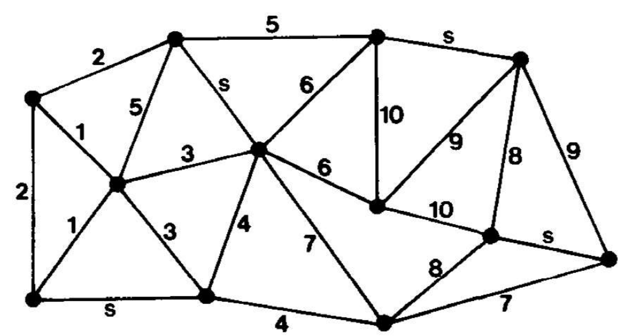
$$3(\text{the number of paths}) = 3(3k) = 9k.$$

But G has only $8k$ vertices, a CONTRADICTION. So the assumption that G has a decomposition into paths of length 4 is false, and hence the claim follows. □

Theorem 2.4.2

Theorem 2.4.2. A connected graph is decomposable into subgraphs each isomorphic to a path of length two if and only if the graph has an even number of edges.

Proof. First, suppose G is a connected graph with an even number of edges. We select as many disjoint paths of length 2 as possible and number the edges of the paths, similar to the following:



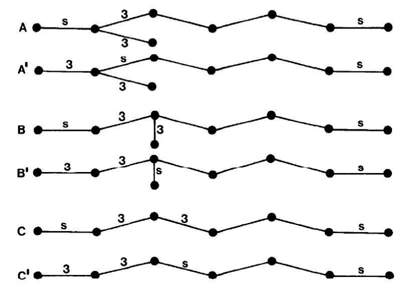
Theorem 2.4.2 (continued 1)

Theorem 2.4.2. A connected graph is decomposable into subgraphs each isomorphic to a path of length two if and only if the graph has an even number of edges.

Proof (continued). If all edges have been used in the paths of length two, the result holds. If some edges are not in the collection of paths of length two, then there will be an even number of single edges left; label then with an s. Since G is connected, there is a path between any pair of single edges marked with an s (this follows from the definition of "connected"). For 2 given edges marked with an s, select a path so that a path between them has shortest length. The first edge (following an edge marked s) of the connecting path belongs to a numbered path of length 2, without loss of generality suppose the path is marked 3. There are three possibilities for the location of the other edge marked 3, as shown below and labeled A, B, and C...

Theorem 2.4.2 (continued 2)

Proof (continued).



We re-mark the edges as given in A', B', and C'. In each case the path between the pair of edges marked s has decreased and all existing paths of length 2 are preserved. This process can be iterated until the two edges marked s form a path of length 2. Performing this process for all remaining pairs of edges marked s allows us to create a decomposition of G into paths of length 2, as claimed. □