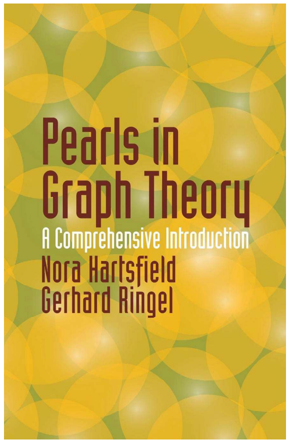
Introduction to Graph Theory

Chapter 3. Circuits and Cycles

3.1. Eulerian Circuits—Proofs of Theorems



Theorem 3.1.1

Theorem 3.1.1. Euler's Theorem.
If a pseudograph G has an Eulerian circuit, then G is connected and the degree of every vertex is even.

Proof. Let $A_1e_1A_2e_2A_3 \cdots A_{n-1}e_{n-1}A_n$ be an Eulerian circuit in G. So there is a walk (and hence a path) between any two vertices of G and G is connected, as claimed. Then the vertices $A_2, A_3, \ldots, A_{n-1}$ are ends of edges in the Eulerian circuit two at a time. Suppose vertex A_i, where $2 \le i \le n-1$, occurs h times in the Eulerian circuit. Since the edges of an Eulerian circuit are distinct, then such vertex A_i is of even degree $2h$. Finally, vertex $A_1 = A_n$ may equal some vertex A_i where $2 \le i \le n-1$. Let $h-1$ be the number of times that A_1 equals such A_i. As just argued, this gives $2(h-1)$ edges incident to A_1. But A_1 also has edges e_1 and e_{n-1} incident to it, so that vertex A_1 is also even degree $2h$. That is, the degree of every vertex of G is even, as claimed. □

Lemma 3.1.3

Lemma 3.1.3. If every vertex in a pseudograph G has positive even degree, then any given vertex of G lies on some circuit of G.

Proof. Recall that a trail and a circuit can have repeated vertices, but not repeated edges. Let A be a vertex in G. If there is a loop with A as its ends then this gives a circuit of length 1. Otherwise, we can create a trail Ae_1A_2 from A to a vertex A_2 adjacent to A. Similarly, we can extend the trail to $Ae_1A_2e_2A_3$ (since A_2 is of even degree) where A_3 is a neighbor of A_2. Inductively, we can extend the trail to $Ae_1A_2e_3A_3 \cdots A_{k-1}e_{k-1}A_k$ based on the even degree of each vertex. Unless $A_k = A$, then vertex A_k has $2h+1$ edges of the trail incident to it (where h is the number of times vertex A_k appears in the trail before it appears as the kth vertex). Since A_k is of even degree, then there is an edge of G incident to A_k which does not appear in the trail, and the trail can be extended. Since G is finite, then the trail must return to vertex A at some stage, giving a circuit containing vertex A, as claimed. □

Theorem 3.1.2

Theorem 3.1.2. Hierholzer's Theorem.
If a pseudograph G is connected and the degree of every vertex of G is even, then G has an Eulerian circuit.

Proof. We give a proof by contradiction (as opposed to a constructive proof). Let G be a connected pseudograph such that the degree of every vertex of G is even. Let C be a longest circuit in G. If C contains every edge of G, then C is an Eulerian circuit, and the claim holds. ASSUME C does not contain every edge of G. Let H be the pseudograph that results by removing all edges of C from G. Since all vertices of G are of even degree by hypothesis, and all vertices of a circuit of of even degree in the circuit, then all vertices of H must be of even degree. Since G is connected, then H and C have a vertex A in common because (or else there would be no edges with one end in $V(H)$ and one end in $V(C)$, so that G does not contain a path between a vertex of $V(H)$ and a vertex of $V(C)$, contradicting the hypothesis of connectivity of G).

Theorem 3.1.2 (continued 1)

Proof (continued). By Lemma 3.1.3, vertex A lies on a circuit C_1 in the component of H that contains A. Then A is a vertex on both circuit C and circuit C_1. Say C is $A_1e_1A_2e_2A_3 \cdots A_{k-1}e_{k-1}Ae_kA_{k+1} \cdots A_{n-1}e_nA_n$ where $A_n = A_1$, and C_1 is $A'_1e'_1A'_2e'_2A'_3 \cdots A'_{k-1}e'_{k-1}Ae'_kA'_{k+1} \cdots A'_{n-1}e'_nA'_n$ where $A'_n = A'_1$. Then we can extend circuit C at A by inserting C_1 to get circuit

$$C' = A_1e_1A_2e_2A_3 \cdots A_{k-1}e_{k-1}Ae'_kA'_{k+1} \cdots A'_{n-1}e'_nA'_ne'_1A'_2e'_2A'_3$$

$$\cdots A'_{k-1}e'_{k-1}Ae_kA_{k+1} \cdots A_{n-1}e_nA_n.$$

But then C' is a circuit in G which is longer than circuit C, a CONTRADICTION to the fact that C was chosen to be the longest circuit in G. So the assumption that C does not contain all edges of G is false, and we have that C is an Eulerian circuit in G, as claimed. □

Note. Hartsfield and Ringel illustrate Hierholzer's construction of C' as given on the next slide.

Theorem 3.1.2 (continued 2)

Theorem 3.1.2. Hierholzer's Theorem.
If a pseudograph G is connected and the degree of every vertex of G is even, then G has an Eulerian circuit.

Proof (continued).

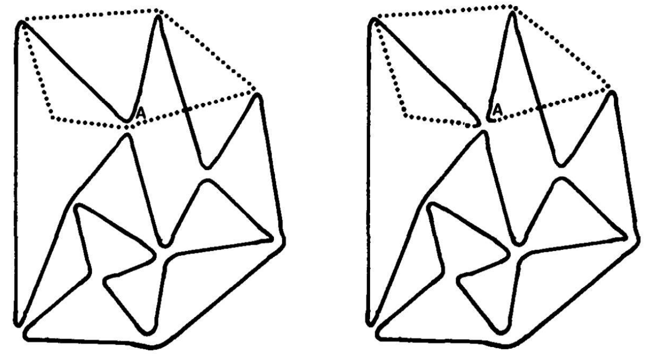


Figure 3.1.4

Theorem 3.1.4

Theorem 3.1.4. If a pseudograph G is regular of degree 4, then G has a decomposition into two 2-factors.

Proof. Let G be a pseudograph that is regular of degree 4. Without loss of generality, we may assume that G is connected, otherwise we could consider each connected component separately. Since every vertex of G has even degree, then G has an Eulerian circuit by Theorem 3.1.2 (Hierholzer's Theorem). With q as the number of edges in G, the length of the Eulerian circuit is q. With p as the number of vertices, the hypothesis regular of degree 4 implies that there are $q = 4p/2 = 2p$ edges in G, so that the Eulerian circuit contains an even number of edges.

Color the edges of the circuit, alternating red and blue. Since the Eulerian circuit must contain each vertex twice (because each vertex is degree 4).

Theorem 3.1.4 (continued)

Theorem 3.1.4. If a pseudograph G is regular of degree 4, then G has a decomposition into two 2-factors.

Proof (continued). The circuit must contain an even number of edges between consecutive appearances of a given vertex in the circuit, so that if a given vertex in the circuit is followed by an edge of color red/blue then when the circuit next returns to the vertex it must be preceded by an edge of color blue red, respectively. So every vertex is incident with two red edges and two blue edges in the Eulerian circuit. The red edges form a 2-factor of G and the blue edges form a second 2-factor of G, as claimed. □

Theorem 3.1.5

Theorem 3.1.5. Veblen's Theorem.
A pseudograph G has a decomposition into cycles if and only if every vertex of G has even degree.

Proof. Suppose G has a decomposition into cycles. Consider an arbitrary vertex A of G. If A belongs to h of these cycles, then A is of even degree $2h$. Therefore, all vertices of G are of even degree.

Now suppose that every vertex of G has even degree. We give an inductive proof on the number of edges in G. For the base case, if G has all even degree vertices and only one or two edges, then the result "clearly" holds. For the induction hypothesis, suppose that the theorem is true for all pseudographs with fewer than n edges and all vertices of even degree. Let G be a pseudograph with n edges and all vertices of even degree. By Lemma 3.1.3, G contains a circuit.

Theorem 3.1.5 (continued)

Theorem 3.1.5. Veblen's Theorem.
A pseudograph G has a decomposition into cycles if and only if every vertex of G has even degree.

Proof (continued). Let C be the shortest circuit in G. Then C must be a cycle, or we could shorten it be deleting all edges repeated between repeated vertices (details are to be given in Exercise 3.1.A). Now consider the pseudograph that results by removing all edges of C from G, which we denote as $G - C$. Since each vertex of G is of even degree by hypothesis and each vertex of C (a cycle) is of degree two, then each vertex of $G - C$ is of even degree and $G - C$ has fewer than n edges. So by the induction hypothesis, $G - C$ has a decomposition into cycles. This collection of cycles which form a decomposition of $G - C$, along with cycle C, yields a decomposition of G into cycles, as claimed. □

Theorem 3.1.6

Theorem 3.1.6. A pseudograph G has an Eulerian trail if and only if G is connected and has precisely two vertices of odd degree.

Proof. Suppose G has an Eulerian trail T. Then there is a trail between any two vertices of G, and so there is a path between any to vertices of G; that is, G is connected (by definition or "connected"). Let the trail begin at vertex A and end at vertex B. Add an edge e between vertices A and B, creating the pseudograph denoted $G + e$. Then $T + e$ is an Eulerian circuit in the pseudograph $G + e$ and, by Theorem 3.1.1, every vertex of $G + e$ must have even degree. Since all vertices of G and $G + e$ have the same degrees, except for A and B, all vertices of G must be of even degree, except for A and B. The degrees of A and B are one larger in $G + e$ than they are in G, so the degrees of A and B must be odd in G. That is, G has exactly two vertices of odd degree, namely A and B.

Theorem 3.1.6 (continued)

Theorem 3.1.6. A pseudograph G has an Eulerian trail if and only if G is connected and has precisely two vertices of odd degree.

Proof (continued). Now suppose that G is connected and has exactly to vertices A and B of odd degree. Again, add an edge e between A and B, creating pseudograph $G + e$. Then $G + e$ is connected, and every vertex has even degree. By Theorem 3.1.2 (Hierholzer's Theorem) $G + e$ has an Eulerian circuit C. Then $C - e$ is an Eulerian trail in G which starts and ends at the vertices of odd degree, as claimed. □

Theorem 3.1.7

Theorem 3.1.7. Listing's Theorem.
If G is a connected pseudograph with precisely $2h$ vertices of odd degree, $h \neq 0$, then there exists h trails in G such that each edge of G is exactly one of these trails. Furthermore, fewer than h trails with this property cannot be found.

Proof. For the $2h$ vertices of odd degree in G, add h new edges in such a way that a pseudograph H is obtained that has only vertices of even degree; that is, edges are added to distinct pairs of degree one vertices. Then H is an Eulerian circuit by Theorem 3.1.2 (Hierholzers Theorem). Next, remove the h added edges from the Eulerian circuit. Since no two of these edges is incident to the same vertex of G, then when these edges are removed from the Eulerian circuit it broken into h trails. Each edge of G is in exactly one of the trails, as claimed. Finally in a collection of trails in G that contain every edge of G, each of the $2h$ vertices of G of odd degree must be the end of a trail. Therefore, such a collection must contain at least h trials, as claimed. □