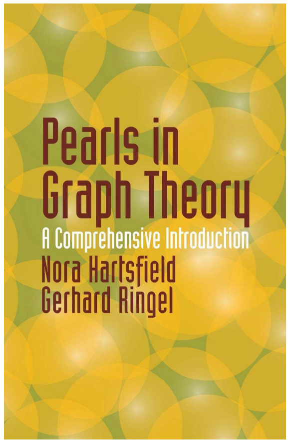
Introduction to Graph Theory

Chapter 3. Circuits and Cycles

3.2. The Oberwolfach Problem—Proofs of Theorems



Theorem 3.2.1

Theorem 3.2.1. A regular graph of even degree has no bridge.

Proof. Let graph G be regular of degree $2h$. ASSUME that b is a bridge of G. Then bridge B determines two banks in $G - b$, and each bank has exactly one vertex of odd degree $2h - 1$ (namely the ends of bridge b). But the number of vertices of a graph of odd degree is even (by Exercise 1.1.5), so this is a CONTRADICTION. So the assumption that regular even degree graph G has a bridge is false, and the claim holds. □

Theorem 3.2.2

Theorem 3.2.2. A cubic graph that contains a bridge is not decomposable into three 1-factors.

Proof. Let G be a cubic graph that contains a bridge b. ASSUME that G is decomposable into three 1-factors. Then b is one of the 1-factors; delete the edges of this 1-factor from G. Each of the remaining 1-factors then induces a 1-factor of each bank of the bridge. Now the number of vertices of a graph of odd degree is even (again, by Exercise 1.1.5), so there must be an odd number of vertices in each bank (the end of bridge b is of even degree in each bank and the remaining vertices in each bank are of degree three). However, a graph with an odd number of vertices (in this case, each bank of the bridge) cannot admit a 1-factor, a CONTRADICTION. So the assumption that G is decomposable inte three 1-factors is false, as claimed. □

Theorem 3.2.4

Theorem 3.2.4. Every cubic bridgeless graph is decomposable into paths of length three.

Proof. Let G be a cubic bridgeless graph. By Petersen's Theorem (Theorem 3.2.3), G has a decomposition into a 1-factor and a 2-factor. Color the edges of the 1-factor blue and the edges of the 2-factor red. Then every vertex is incident with one blue edge and two red edges. Number the blue edges $1, 2, \ldots, \ell$ in any order. By Theorem 3.1.5, a 2-factor is a collection of (red) cycles and, since G is cubic, the cycles are pairwise edge disjoint. Imagine traveling around a red cycle (this gives an orientation to the cycle). We then number the red edges of the cycle by the same number as the blue edge incident with the *beginning* of that red edge. This is illustrated in Figure 3.2.6 in the next slide. More formally, let a red cycle be described as an alternating sequence of distinct vertices and distinct edges as $A_1 e_1 A_2 e_2 A_3 \cdots A_\ell e_\ell A_1$. Each vertex $A_1, A_2, \ldots, A_\ell$ has a blue edge incident to it; say blue edge f_i is incident to vertex A_i. Assign to red edge e_i the same number as assigned to edge f_i, for $1 \le i \le \ell$.

Theorem 3.2.4 (continued)

Proof (continued).

(In Figure 3.2.6, the edges drawn as straight lines are blue and the edges drawn as curves are red edges.) Every edge is now labeled in such a way that the three edges with the same number form a path of length three (a red edge, followed by a blue edge, followed by a different red edge). So G has a decomposition into paths of length three, as claimed.

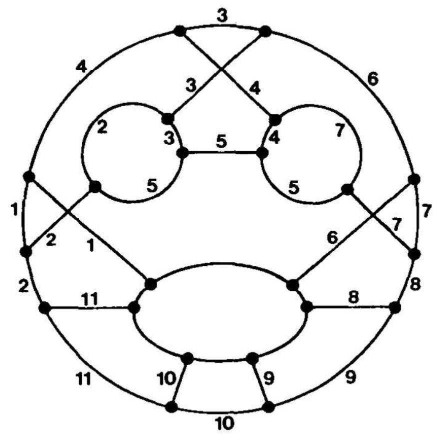


Figure 3.2.6

□