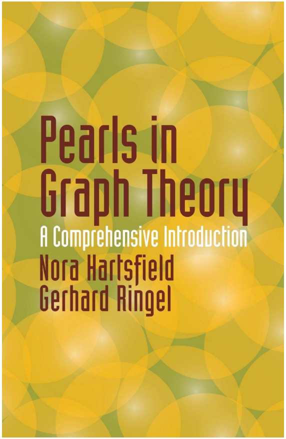
Introduction to Graph Theory

Chapter 3. Circuits and Cycles

3.2. The Oberwolfach Problem—Proofs of Theorems



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Theorem 3.2.1

Theorem 3.2.1. A regular graph of even degree has no bridge.

Proof. Let graph G be regular of degree $2h$. ASSUME that b is a bridge of G. Then bridge B determines two banks in $G - b$, and each bank has exactly one vertex of odd degree $2h - 1$ (namely the ends of bridge b). But the number of vertices of a graph of odd degree is even (by Exercise 1.1.5), so this is a CONTRADICTION. So the assumption that regular even degree graph G has a bridge is false, and the claim holds. □

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Theorem 3.2.2

Theorem 3.2.2. A cubic graph that contains a bridge is not decomposable into three 1-factors.

Proof. Let G be a cubic graph that contains a bridge b. ASSUME that G is decomposable into three 1-factors. Then b is one of the 1-factors; delete the edges of this 1-factor from G. Each of the remaining 1-factors then induces a 1-factor of each bank of the bridge.

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Theorem 3.2.4

Theorem 3.2.4. Every cubic bridgeless graph is decomposable into paths of length three.

Proof. Let G be a cubic bridgeless graph. By Petersen's Theorem (Theorem 3.2.3), G has a decomposition into a 1-factor and a 2-factor. Color the edges of the 1-factor blue and the edges of the 2-factor red. Then every vertex is incident with one blue edge and two red edges. Number the blue edges $1, 2, \ldots, \ell$ in any order. By Theorem 3.1.5, a 2-factor is a collection of (red) cycles and, since G is cubic, the cycles are pairwise edge disjoint. Imagine traveling around a red cycle (this gives an orientation to the cycle). We then number the red edges of the cycle by the same number as the blue edge incident with the *beginning* of that red edge. This is illustrated in Figure 3.2.6 in the next slide.

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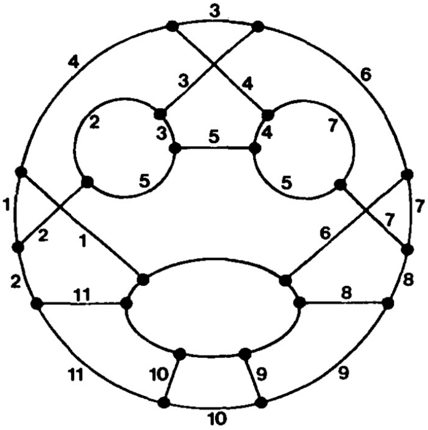
Theorem 3.2.4 (continued)

Proof (continued).

(In Figure 3.2.6, the edges drawn as straight lines are blue and the edges drawn as curves are red edges.) Every edge is now labeled in such a way that the three edges with the same number form a path of length three (a red edge, followed by a blue edge, followed by a different red edge). So G has a decomposition into paths of length three, as claimed.



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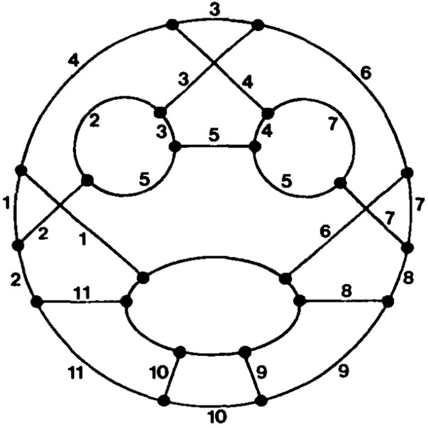

Figure 3.2.6

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