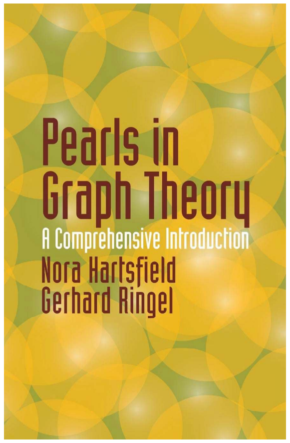
# Introduction to Graph Theory

**Chapter 4. Extremal Problems**

4.1. A Theorem of Turan—Proofs of Theorems








## Theorem 4.1.A

**Theorem 4.1.A.** The largest graph $G$ (that is, with the most edges) with chromatic number two and $n$ vertices is a complete bipartite graph $K_{n_1,n_2}$ where $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.

**Proof.** Let graph $G$ have chromatic number two based on the colors, say, red and blue, and let $G$ be a largest such graph on $n$ vertices. If a red vertex is not adjacent to a blue vertex, then an edge can be added joining these two vertices (increasing the number of edges). So in a largest graph, every blue vertex is adjacent to every red vertex and so a largest such graph is a complete bipartite graph (of course, no two red vertices are adjacent and no two blue vertices are adjacent). Suppose there are $n_1$ blue vertices and $n_2$ red vertices so that $n_1 + n_2 = n$. Suppose that $n_1 \le n_2$.





## Theorem 4.1.A (continued)

**Theorem 4.1.A.** The largest graph $G$ (that is, with the most edges) with chromatic number two and $n$ vertices is a complete bipartite graph $K_{n_1,n_2}$ where $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.

**Proof (continued).** ASSUME $n_1 + 2 \le n_2$, then we can create another complete bipartite graph $\hat{G}$ with $n_1 + 1$ blue vertices and $n_2 - 1$ red vertices. Then $G$ has $n_1 n_2$ edges, and $\hat{G}$ has $(n_1 + 1)(n_2 - 1)$ edges. We have $(n_1 + 1)(n_2 - 1) - n_1 n_2 = n_2 - n_1 - 1$. Since we assumed $n_2 - n_1 \ge 2$ then $n_2 - n_1 - 1 \ge 1$, and hence $\hat{G}$ has at least one more edge than $G$, still has chromatic number two, and has $n$ vertices. But this is a CONTRADICTION to the fact that $G$ is a largest chromatic number two graph with $n$ vertices. So the assumption that $n_1 + 2 \le n_2$ is false (and hence $n_1 + 2 > n_2$) and (since $n_1 \le n_2$) we must have $n_2 - n_1 = 0$ or $n_2 - n_1 = 1$; that is, $|n_1 - n_2| \le 1$. Since $n_1 + n_2 = n$, then we must have $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$, as claimed. □





## Theorem 4.1.1

**Theorem 4.1.1.** The largest graph $G$ (that is, with the most edges) with chromatic number $k$ and $n$ vertices is a complete $k$-partite graph $K_{n_1,n_2,\dots n_k}$ where $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \le 1$.

**Proof.** Let graph $G$ have chromatic number $k$ based on the colors, say, $i = 1, 2, \dots, k$, and let $G$ be a largest such graph on $n$ vertices. Let $m_i$ be the number of vertices colored with color $i$. Then $n = n_1 + n_2 + \cdots + n_k$. Since $G$ has a maximum number of edges, every pair of vertices that are colored with different colors is adjacent. So $G$ is a complete $k$-partite graph $K_{n_1,n_2,\dots,n_k}$. Let the number of edges in a $k$-partite graph $K_{n_1,n_2,\dots,n_k}$ be denoted by $A(n_1, n_2, \dots, n_k)$. Then we claim $A(n_1, n_2, n_3, \dots, n_k) = n_1 n_2 + A(n_1 + n_2, n_3, \dots, n_k)$. This holds because on the right-hand side $A(n_1 + n_2, n_3, \dots, n_k)$ are the edges in a $(k-1)$-partite graph, where we have combined the first two partite sets of $K_{n_1,n_2,\dots,n_k}$ into a single partite set (and so we have lost the $n_1 n_2$ edges between the first two partite sets).

## Theorem 4.1.1 (continued 1)

**Theorem 4.1.1.** The largest graph $G$ (that is, with the most edges) with chromatic number $k$ and $n$ vertices is a complete $k$-partite graph $K_{n_1,n_2,\dots n_k}$ where $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \le 1$.

**Proof (continued).** ASSUME that in $G$ any two of the numbers $n_1, n_2, \dots, n_k$ differ by more that one, say (without loss of generality) $n_1 + 2 \le n_2$ where $n_1 \le n_2$. Then we can create another complete $k$-partite graph $\hat{G} = K_{n_1+1,n_2-1,n_3,\dots,n_k}$. The number of edges in $\hat{G}$ is then $A(n_1+1, n_2-1, n_3, \dots, n_k) = (n_1+1)(n_2-1) + A(n_1+n_2, n_3, \dots, n_k)$. The number of edges in $\hat{G}$ minus the number of edges in $G$ is $(n_1+1)(n_2-1) - n_1 n_2 = n_2 - n_1 - 1$, because there are the same number of edges between the first two partite sets and the other partite sets in both $G$ and $\hat{G}$, but between the first two partite sets there at $(n_1+1)(n_2-1)$ edges in $\hat{G}$ and $n_1 n_2$ edges edges in $G$.





## Theorem 4.1.1 (continued 2)

**Theorem 4.1.1.** The largest graph $G$ (that is, with the most edges) with chromatic number $k$ and $n$ vertices is a complete $k$-partite graph $K_{n_1,n_2,\dots n_k}$ where $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \le 1$.

**Proof (continued).** Since $n_2 - n_1 \ge 2$ we have $n_2 - n_1 - 1 \ge 1$, $\hat{G}$ has at least one more edge then $G$, CONTRADICTING the fact that $G$ is a largest chromatic number $k$ graph on $n$ vertices. So the assumption that $n_1 + 2 \le n_2$ is false (and hence $n_1 + 2 > n_2$) and (since $n_1 \le n_2$) we must have $n_2 - n_1 = 0$ or $n_2 - n_1 = 1$; that is, $|n_1 - n_2| \le 1$. Since the result holds for any $n_1$ and $n_2$, we conclude that $K_{n_1,n_2,\dots n_k}$ where $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \le 1$ for all $1 \le i, j \le k$, as claimed. □

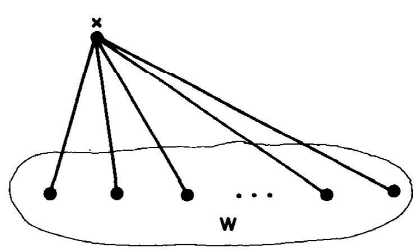




## Theorem 4.1.2*

**Theorem 4.1.2*.** The largest graph (that is, with the most edges) with $n$ vertices that contains no triangle is the complete bipartite graph $K_{n_1,n_2}$ with $n = n_1 + n_2$ and $|n_1 - n_2| \le 1$.

**Proof.** Let $G$ be a graph with $n$ vertices that does not contain a triangle, and let $V$ be the vertex set of $G$. Let $x$ be a vertex of $G$ with the largest degree in $G$; that is, $\deg_G(x)$ is maximal. Consider the set $W$ of vertices of $G$ that are adjacent to $x$ ($W$ is often called the *neighborhood* of $x$). No two vertices in $W$ can be adjacent, since this would yield a triangle in $G$.

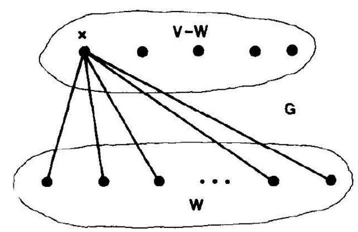


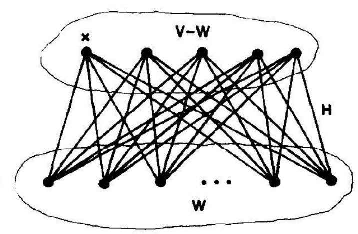





## Theorem 4.1.2* (continued 1)

**Proof (continued).** Define a new graph $H$ with vertex set $V$, and let $W$ be the same subset of $V$ as above. Let $H$ be the complete bipartite graph with all edges joining elements of $V - W$ to elements of $W$.




If $z$ is a vertex in $V - W$, then $\deg_H(z) = \deg_H(x) = \deg(x) \ge \deg_G(z)$, since we chose $x$ in $G$ as a vertex of maximal degree in $G$.

## Theorem 4.1.2* (continued 2)

**Theorem 4.1.2*.** The largest graph (that is, with the most edges) with $n$ vertices that contains no triangle is the complete bipartite graph $K_{n_1,n_2}$ with $n = n_1 + n_2$ and $|n_1 - n_2| \leq 1$.

**Proof (continued).** Let the number of vertices in $W$ be $w$. Then if $z$ is a vertex in $W$, $\deg_H(z) = n - w \geq \deg_G(z)$, since no two vertices in $W$ could be adjacent in $G$ (because $G$ contains no triangle). So in graph $H$, every vertex $z$ in $G$ satisfies $\deg_G(z) \leq \deg_H(z)$. So the total number of edges in $H$ must be at least the number of edges in $G$. Since $G$ is an arbitrary graph with $n$ vertices that does not contain a triangle, then we see that $H$ is the largest such graph. Since $H$ is a complete bipartite graph with partite sets of sizes, say, $n_1$ and $n_2$ (notice that $w \in \{n_1, n_2\}$ is the maximum degree of a vertex in $G$ and in $H$), and $H$ is a largest such bipartite graph, then we have from the proof of Theorem 4.1.1 that $H$ must be of the form $H = K_{n_1,n_2}$ where $n = n_1 + n_2$ and $|n_1 - n_2| \leq 1$, as claimed. □

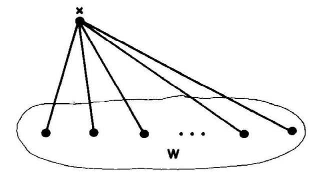




## Lemma 4.1.3

**Lemma 4.1.3.** If $G$ is a graph on $n$ vertices that contains no $K_{k+1}$ then there is a $k$-partite graph $H$ with the same vertex set as $G$ such that $\deg_G(z) \leq \deg_H(z)$ for every vertex $z$ of $G$.

**Proof.** We give an inductive proof on $k \geq 2$. The base case $k = 2$ is given in the proof of Theorem 4.1.2*. For the induction hypothesis, suppose that the lemma holds for all values less than $k$.



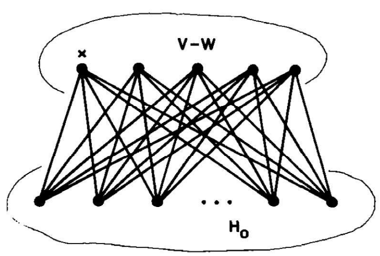

Let $G$ be a graph with $n$ vertices that does not contain a $K_{k+1}$. Let $V$ be the vertex set of $G$, and let $x$ be a vertex of $G$ with $\deg_G(x)$ maximum. Let $W$ be the set of vertices adjacent to $x$ in $G$ (i.e., $W$ is the neighborhood of $x$), and let $G_0$ be the subgraph of $G$ induced by the set $W$. Now $G_0$ cannot contain a $K_k$, otherwise $G$ would contain a $K_{k+1}$ since $x$ is adjacent to all vertices in $W$.





## Lemma 4.1.3 (continued 1)

**Proof (continued).** So by the induction hypothesis, the lemma holds for $G_0$, and so there exists a $(k-1)$-partite graph $H_0$ such that $\deg_{G_0}(z) \leq \deg_{H_0}(z)$ for every vertex $z$ in $W$. Next, connect each vertex in $V - W$ to every vertex in $H_0$ to form graph $H$.




For every vertex $z$ in $V - W$ we have (in graph $G$) $\deg_G(z) \leq \deg_G(x)$, since $x$ has maximum degree in graph $G$, and $\deg_G(x) = \deg_H(x) = \deg_H(z)$ since every vertex in $V - W$ is connected to every vertex in $W$ in graph $H$. So for every vertex $z$ in $V - W$ we have

$$\deg_G(z) \leq \deg_H(z). \qquad (*)$$





## Lemma 4.1.3 (continued 2)

**Proof (continued).** Now let $z$ be a vertex in $W$, and let $w$ be the number of vertices in $W$. Then $\deg_G(z) \leq \deg_{G_0}(z) + n - w$ since $z$ can be adjacent to at most all of the $n - w$ vertices in $V - W$ in graph $G$; recall that $G_0$ is the subgraph of $G$ induced by $W$ and so includes all edges of $G$ between vertices in $W$). Also, since $\deg_{G_0}(z) \leq \deg_{H_0}(z)$ for every vertex $z$ in $W$ as shown above (by the induction hypothesis) then $\deg_{G_0}(z) + n - w \leq \deg_{H_0}(z) + n - w$ for every vertex $z$ in $W$. Now $\deg_{H_0}(z) + n - w = \deg_H(z)$ for every vertex $z$ in $W$, since $H_0$ is a graph on $W$ and $H$ results from connecting each vertex of $H$ to all vertices in $V - W$. Therefore, for all vertices $z$ in $W$ we have

$$\deg(z) \leq \deg_{G_0}(z) + n - w \leq \deg_{H_0}(z) + n - w \leq \deg_H(z). \qquad (**)$$

Hence, combining $(*)$ and $(**)$, we have $\deg_G(z) \leq \deg_H(z)$ for every vertex $z$ in $G$, as claimed. Therefore, by mathematical induction on $k$, the lemma holds for all $k$. □

# Theorem 4.1.2. Turan's Theorem

**Theorem 4.1.2. Turan's Theorem.**
The largest graph (that is, with the most edges) with $n$ vertices that contains no subgraph isomorphic to $K_{k+1}$ is a complete $k$-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \le 1$.

**Proof.** Let $G$ be a graph with $n$ vertices that does not contain a subgraph isomorphic to $K_{k+1}$. Then by Lemma 4.1.3, $G$ can be used to construct a $k$-partite graph (denoted $H$ in Lemma 4.1.3) without decreasing the number of edges. By Theorem 4.1.1, the largest $k$-partite graph with $n$ vertices is the complete $k$-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $|n_i - n_j| \le 1$. So from given graph $G$, we can construct graph $H$ and then add edges as described in Theorem 4.1.1 until we have the largest such $k$-partite graph, with the structure as claimed. □