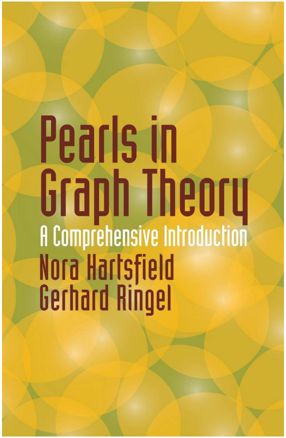
Introduction to Graph Theory

Chapter 4. Extremal Problems

4.1. A Theorem of Turan—Proofs of Theorems



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Theorem 4.1.A

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Proof. Let graph G have chromatic number two based on the colors, say, red and blue, and let G be a largest such graph on n vertices. If a red vertex is not adjacent to a blue vertex, then an edge can be added joining these two vertices (increasing the number of edges). So in a largest graph, every blue vertex is adjacent to every red vertex and so a largest such graph is a complete bipartite graph (of course, no two red vertices are adjacent and no two blue vertices are adjacent). Suppose there are n_1 blue vertices and n_2 red vertices so that $n_1 + n_2 = n$. Suppose that $n_1 \leq n_2$.

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Proof (continued). ASSUME $n_1 + 2 \leq n_2$, then we can create another complete bipartite graph $\hat{G}$ with $n_1 + 1$ blue vertices and $n_2 - 1$ red vertices. Then G has $n_1 n_2$ edges, and $\hat{G}$ has $(n_1 + 1)(n_2 - 1)$ edges. We have $(n_1 + 1)(n_2 - 1) - n_1 n_2 = n_2 - n_1 - 1$. Since we assumed $n_2 - n_1 \geq 2$ then $n_2 - n_1 - 1 \geq 1$, and hence $\hat{G}$ has at least one more edge than G, still has chromatic number two, and has n vertices. But this is a CONTRADICTION to the fact that G is a largest chromatic number two graph with n vertices. So the assumption that $n_1 + 2 \leq n_2$ is false (and hence $n_1 + 2 > n_2$) and (since $n_1 \leq n_2$) we must have $n_2 - n_1 = 0$ or $n_2 - n_1 = 1$; that is, $|n_1 - n_2| \leq 1$. Since $n_1 + n_2 = n$, then we must have $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$, as claimed. □

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Proof. Let graph G have chromatic number k based on the colors, say, $i = 1, 2, \ldots, k$, and let G be a largest such graph on n vertices. Let m_i be the number of vertices colored with color i. Then $n = n_1 + n_2 + \cdots + n_k$. Since G has a maximum number of edges, every pair of vertices that are colored with different colors is adjacent. So G is a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$.

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Proof (continued). ASSUME that in G any two of the numbers $n_1, n_2, \ldots, n_k$ differ by more that one, say (without loss of generality) $n_1 + 2 \leq n_2$ where $n_1 \leq n_2$. Then we can create another complete k-partite graph $\hat{G} = K_{n_1+1,n_2-1,n_3,\ldots,n_k}$. The number of edges in $\hat{G}$ is then $A(n_1+1, n_2-1, n_3, \ldots, n_k) = (n_1+1)(n_2-1) + A(n_1+n_2, n_3, \ldots, n_k)$. The number of edges in $\hat{G}$ minus the number of edges in G is $(n_1+1)(n_2-1) - n_1 n_2 = n_2 - n_1 - 1$, because there are the same number of edges between the first two partite sets and the other partite sets in both G and $\hat{G}$, but between the first two partite sets there at $(n_1+1)(n_2-1)$ edges in $\hat{G}$ and $n_1 n_2$ edges edges in G.

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Theorem 4.1.2*

Theorem 4.1.2*. The largest graph (that is, with the most edges) with n vertices that contains no triangle is the complete bipartite graph K_{n_1,n_2} with $n = n_1 + n_2$ and $|n_1 - n_2| \leq 1$.

Proof. Let G be a graph with n vertices that does not contain a triangle, and let V be the vertex set of G. Let x be a vertex of G with the largest degree in G; that is, $\deg_G(x)$ is maximal. Consider the set W of vertices of G that are adjacent to x (W is often called the *neighborhood* of x). No two vertices in W can be adjacent, since this would yield a triangle in G.

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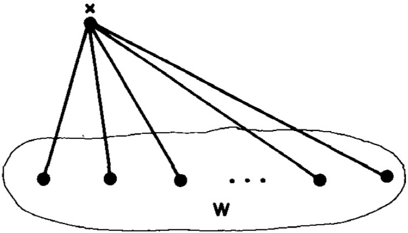

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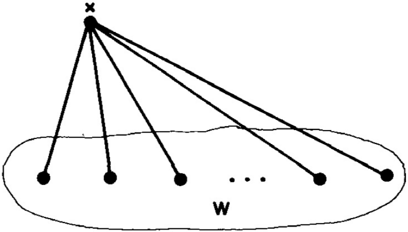
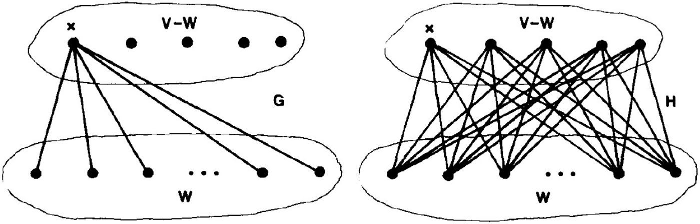


Theorem 4.1.2* (continued 1)

Proof (continued). Define a new graph H with vertex set V, and let W be the same subset of V as above. Let H be the complete bipartite graph with all edges joining elements of $V - W$ to elements of W.



If z is a vertex in $V - W$, then $\deg_H(z) = \deg_H(x) = \deg(x) \geq \deg_G(z)$, since we chose x in G as a vertex of maximal degree in G.

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Lemma 4.1.3

Lemma 4.1.3. If G is a graph on n vertices that contains no K_{k+1} then there is a k-partite graph H with the same vertex set as G such that $\deg_G(z) \leq \deg_H(z)$ for every vertex z of G.

Proof. We give an inductive proof on $k \geq 2$. The base case $k = 2$ is given in the proof of Theorem 4.1.2*. For the induction hypothesis, suppose that the lemma holds for all values less than k.

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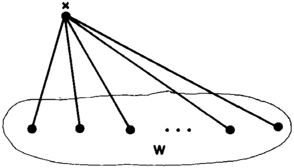

Let G be a graph with n vertices that does not contain a K_{k+1}. Let V be the vertex set of G, and let x be a vertex of G with $\deg_G(x)$ maximum. Let W be the set of vertices adjacent to x in G (i.e., W is the neighborhood of x), and let G_0 be the subgraph of G induced by the set W. Now G_0 cannot contain a K_k, otherwise G would contain a K_{k+1} since x is adjacent to all vertices in W.

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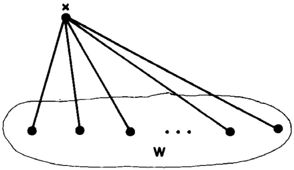

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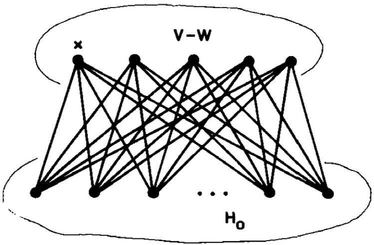
Lemma 4.1.3 (continued 1)

Proof (continued). So by the induction hypothesis, the lemma holds for G_0, and so there exists a $(k-1)$-partite graph H_0 such that $\deg_{G_0}(z) \leq \deg_{H_0}(z)$ for every vertex z in W. Next, connect each vertex in $V - W$ to every vertex in H_0 to form graph H.

For every vertex z in $V - W$ we have (in graph G) $\deg_G(z) \leq \deg_G(x)$, since x has maximum degree in graph G, and $\deg_G(x) = \deg_H(x) = \deg_H(z)$ since every vertex in $V - W$ is connected to every vertex in W in graph H. So for every vertex z in $V - W$ we have



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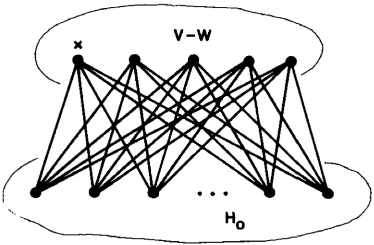

$$\deg_G(z) \leq \deg_H(z). \qquad (*)$$

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Proof (continued). Now let z be a vertex in W, and let w be the number of vertices in W. Then $\deg_G(z) \leq \deg_{G_0}(z) + n - w$ since z can be adjacent to at most all of the $n - w$ vertices in $V - W$ in graph G; recall that G_0 is the subgraph of G induced by W and so includes all edges of G between vertices in W). Also, since $\deg_{G_0}(z) \leq \deg_{H_0}(z)$ for every vertex z in W as shown above (by the induction hypothesis) then $\deg_{G_0}(z) + n - w \leq \deg_{H_0}(z) + n - w$ for every vertex z in W. Now $\deg_{H_0}(z) + n - w = \deg_H(z)$ for every vertex z in W, since H_0 is a graph on W and H results from connecting each vertex of H to all vertices in $V - W$. Therefore, for all vertices z in W we have

$$\deg(z) \leq \deg_{G_0}(z) + n - w \leq \deg_{H_0}(z) + n - w \leq \deg_H(z). \qquad (**)$$

Hence, combining $(*)$ and $(**)$, we have $\deg_G(z) \leq \deg_H(z)$ for every vertex z in G, as claimed. Therefore, by mathematical induction on k, the lemma holds for all k. □

Lemma 4.1.3 (continued 2)

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Theorem 4.1.2. Turan's Theorem

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The largest graph (that is, with the most edges) with n vertices that contains no subgraph isomorphic to K_{k+1} is a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \leq 1$.

Proof. Let G be a graph with n vertices that does not contain a subgraph isomorphic to K_{k+1}. Then by Lemma 4.1.3, G can be used to construct a k-partite graph (denoted H in Lemma 4.1.3) without decreasing the number of edges. By Theorem 4.1.1, the largest k-partite graph with n vertices is the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $|n_i - n_j| \leq 1$. So from given graph G, we can construct graph H and then add edges as described in Theorem 4.1.1 until we have the largest such k-partite graph, with the structure as claimed. □

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