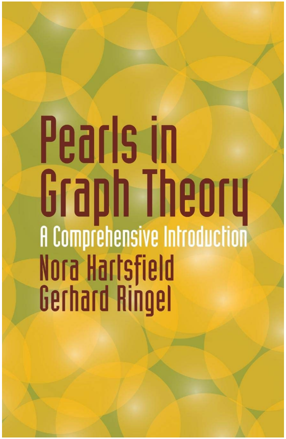
Introduction to Graph Theory

Chapter 4. Extremal Problems

4.2. Cages—Proofs of Theorems



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Theorem 4.2.1

Theorem 4.2.1. The Petersen graph is the unique 5-cage.

Proof. Let G be any 5-cage. We will follow a sequence of general steps and see that this necessarily leads to the Petersen graph. Fix a vertex of G that lies on a 5-cycle and denote it as 1. Since G is cubic, 1 has three neighbors which we denote 2, 3, and 4. Now each of the vertices 2, 3, 4 has two neighbors in addition to 1, as shown in Figure 4.2.1.

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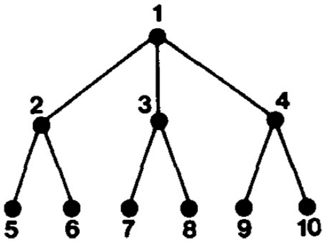

Figure 4.2.1

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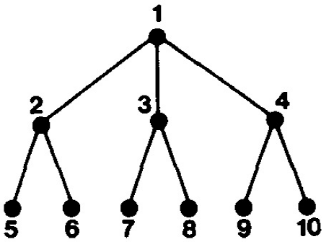

Figure 4.2.1

Theorem 4.2.1 (continued 1)

Theorem 4.2.1. The Petersen graph is the unique 5-cage.

Proof (continued). Denote these vertices as 5, 6, 7, 8, 9, and 10 as given in the figure. None of 5, 6, 7, 8, 9, 10 can be equal to any of 2, 3, or 4 because this would imply that G contains a triangle. Also, all vertices 1, 2, ..., 10 must be distinct, or else G would contain a cycle of length three or four. So we have that a 5-cage must have at least 10 vertices. Since a g-cage is a smallest cubic graph of girth g, then if we can construct a 5-cage on 10 vertices then it is smallest. Now vertex 5 is adjacent to two more vertices and it cannot be adjacent to vertex 6 (or we get a triangle). It then must be adjacent to one of 7 or 8 and one of 9 or 10 (for if it is adjacent to both 7 or 8 then there is a 4-cycle, and similarly if it is adjacent to both 9 and 10). Without loss of generality, say 5 is adjacent to 7 and 9. Similarly, vertex 6 cannot be adjacent to 7 or 9 (or we get a 4-cycle), so 6 must be adjacent to 8 and 10. See the figure below.

Theorem 4.2.1 (continued 1)

Theorem 4.2.1. The Petersen graph is the unique 5-cage.

Proof (continued). Denote these vertices as 5, 6, 7, 8, 9, and 10 as given in the figure. None of 5, 6, 7, 8, 9, 10 can be equal to any of 2, 3, or 4 because this would imply that G contains a triangle. Also, all vertices 1, 2, ..., 10 must be distinct, or else G would contain a cycle of length three or four. So we have that a 5-cage must have at least 10 vertices. Since a g-cage is a smallest cubic graph of girth g, then if we can construct a 5-cage on 10 vertices then it is smallest. Now vertex 5 is adjacent to two more vertices and it cannot be adjacent to vertex 6 (or we get a triangle). It then must be adjacent to one of 7 or 8 and one of 9 or 10 (for if it is adjacent to both 7 or 8 then there is a 4-cycle, and similarly if it is adjacent to both 9 and 10). Without loss of generality, say 5 is adjacent to 7 and 9. Similarly, vertex 6 cannot be adjacent to 7 or 9 (or we get a 4-cycle), so 6 must be adjacent to 8 and 10. See the figure below.

Theorem 4.2.1 (continued 2)

Theorem 4.2.1. The Petersen graph is the unique 5-cage.

Proof (continued). Again, vertex 7 cannot be adjacent to 8 or 9 (or we get a triangle), so 7 must be adjacent to 10. Since G is cubic, 8 and 9 must be adjacent. See Figure 4.2.2.

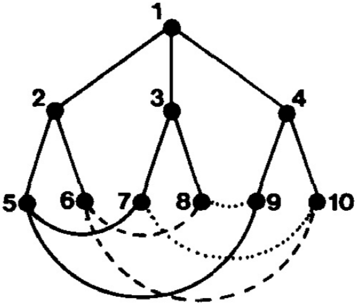

Figure 4.2.2

Theorem 4.2.1 (continued 2)

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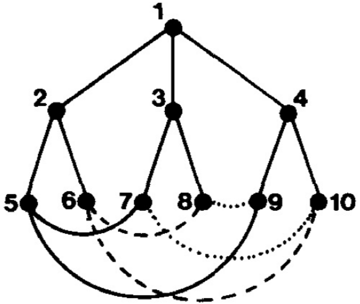

Figure 4.2.2

Theorem 4.2.1 (continued 3)

Proof (continued).

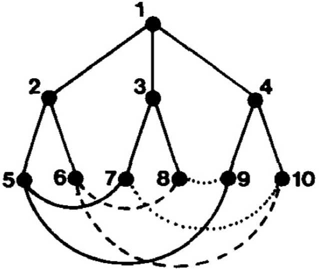

Figure 4.2.2

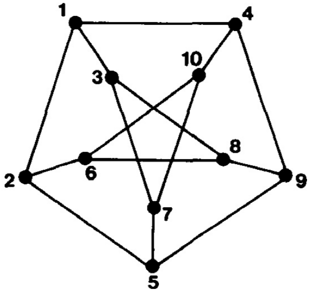

Figure 4.2.3. The Petersen graph.

Since G is a cubic graph of girth 5 on the smallest possible number of vertices (and hence on the smallest number of edges), then it is a 5-cage. Since no restrictions were put on the construction of this 5-cage, then any 5-cage is necessarily isomorphic to G and, as shown in Figure 4.2.3, G is isomorphic to the Petersen graph. $\square$

Theorem 4.2.2

Theorem 4.2.2. The Heawood graph of Figure 4.2.4 is the unique 6-cage.

Proof. Let G be any 6-cage. Fix an edge e of G and label its endpoints 1 and 2. Since G is cubic, each endpoint is adjacent to two distinct vertices, say vertices 3, 4, 5, and 6 as shown in Figure 4.2.5.

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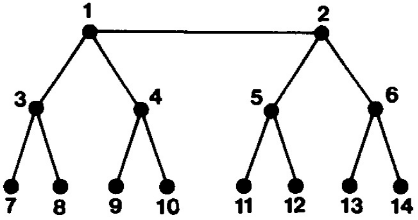

Figure 4.2.5

None of 3, 4, 5, 6 can be adjacent to each other (otherwise G would have a cycle of length 3 or 4), and so these vertices must be adjacent to (in pairs) some vertices 7, 8, 9, 10, 11, 12, 13, 14.

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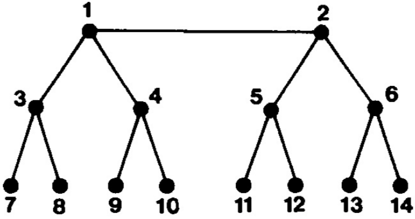

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None of 3, 4, 5, 6 can be adjacent to each other (otherwise G would have a cycle of length 3 or 4), and so these vertices must be adjacent to (in pairs) some vertices 7, 8, 9, 10, 11, 12, 13, 14.

Theorem 4.2.2 (continued 1)

Proof (continued).

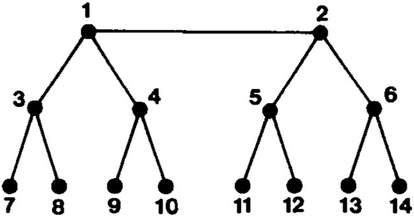


Figure 4.2.5

These new vertices must be distinct (otherwise G would have a cycle of length 4 or 5). So a 6-cage must have at least 14 vertices. Since a g-cage is a smallest cubic graph of girth g, then if we can construct a 6-cage on 14 vertices it is smallest. None of 7, 8, 9, 10 can be adjacent (or else G would contain either a 3-cycle or a 5-cycle); similarly, non of 11, 12, 13, 14 can be adjacent. If 7 is adjacent to both 11 and 12 and on of 13 or 14, say 7 is adjacent to 11 and 13.

Theorem 4.2.2 (continued 2)

Proof (continued).

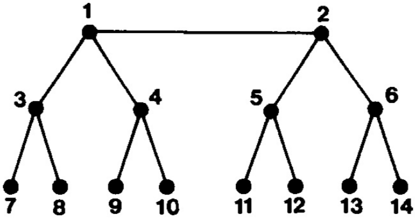
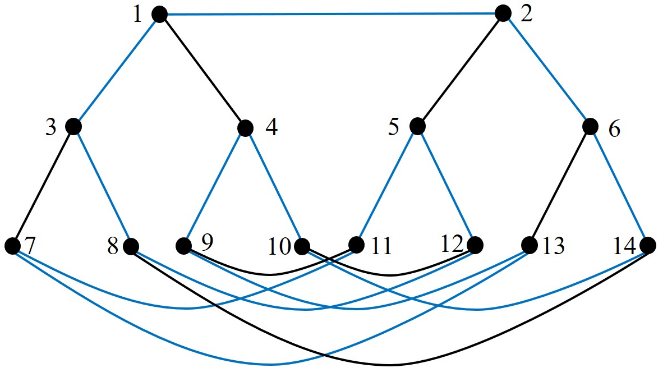


Figure 4.2.5

Vertex 8 has similar constraints on its neighbors, so say 8 is adjacent to 12 and 14 (notice that vertices 7 and 8 cannot share a neighbor in addition to 3, or else G contains a 4-cycle). Similarly, vertex 9 (and vertex 10) must be adjacent to one of 11 or 12 and one of 13 or 14. Say 9 is adjacent to 11 and 13, and 10 is adjacent to 12 and 14, as in the figure below. (There is a symmetry here between the roles of (7 and 8) and (9 and 10), as well as a symmetry between (11 and 12) and (13 and 14).)

Theorem 4.2.2 (continued 3)

Proof (continued). We then have the graph:

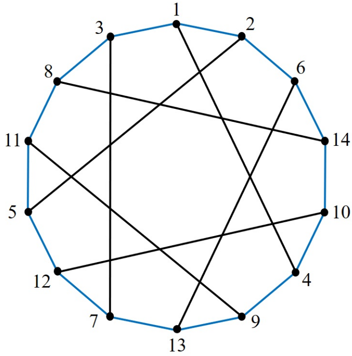


The blue edges are the outer cycle of the Heawood graph.

Theorem 4.2.2 (continued 4)

Theorem 4.2.2. The Heawood graph of Figure 4.2.4 is the unique 6-cage.

Proof (continued).



We see in this figure that the resulting graph is isomorphic to Heawood's graph, as claimed. □