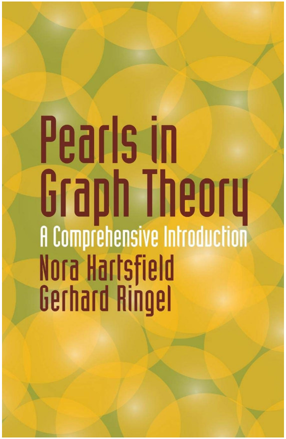
Introduction to Graph Theory

Chapter 4. Extremal Problems

4.3. Ramsey Theory—Proofs of Theorems



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Lemma 4.3.A

Lemma 4.3.A. If the edges of K_6 are colored with two colors, then there must be a monochromatic triangle. Also, K_6 is minimal complete graph with respect to this property.

Proof. Suppose the edges of K_6 are colored red and blue. Let v be any vertex of K_6. Since there are five edges incident to v, then there are either at least three red edges or three blue edges incident to v. Say, without loss of generality, there are three red edges incident to v.

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I any of the dotted edges in Figure 4.3.1 is red, there is a red triangle. If all the dotted edges are blue, then they form a blue triangle. Thus any edge-coloring of K_6 by two colors contains a monochromatic triangle.

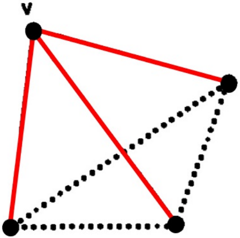

Figure 4.3.1

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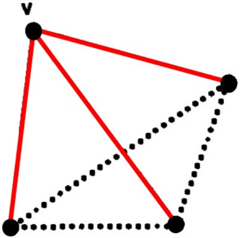

Figure 4.3.1

Lemma 4.3.A (continued)

Proof (continued). To show the minimality of K_6 with respect to the property, we simply need to show that K_5 can be two colored in such a way as to not have a monochromatic triangle (then for the cases of K_4 and K_3, we can simply treat these as subgraphs of the two colored K_5). This is given in Figure 4.3.2. □

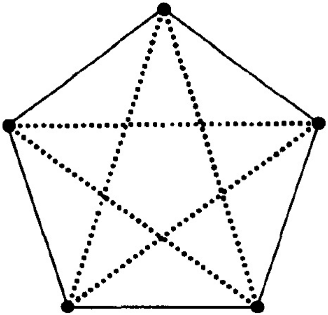

Figure 4.3.2

Lemma 4.3.B

Lemma 4.3.B. If the edges of K_9 are colored with red and blue, then there is a subgraph of this K_9 that is either a red K_3 or a blue K_4. Also, K_9 is minimal complete graph with respect to this property.

Proof. Assume that the edges of K_9 are colored with red and blue. If any vertex of K_9 has four red edges incident to it then claim that the K_9 must contain a red triangle or a blue K_4. See Figure 4.3.3.

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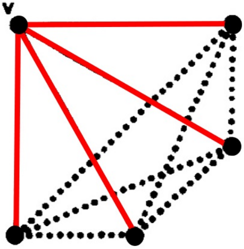
Proof. Assume that the edges of K_9 are colored with red and blue. If any vertex of K_9 has four red edges incident to it then claim that the K_9 must contain a red triangle or a blue K_4. See Figure 4.3.3.

If any of the dotted lines is red, then the graph contains a red triangle.
If all of the dotted lines are blue, then the graph contains a blue K_4.



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Proof. Assume that the edges of K_9 are colored with red and blue. If any vertex of K_9 has four red edges incident to it then claim that the K_9 must contain a red triangle or a blue K_4. See Figure 4.3.3.

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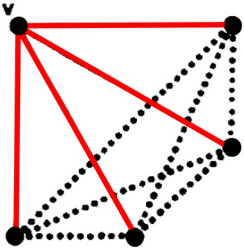

Figure 4.3.3

Lemma 4.3.B (continued 1)

Proof (continued). If some vertex has six blue edges incident with it then claim that the K_9 must contain a red triangle or a blue K_4. For such a vertex, the other ends of these six blue edges determine a K_6 subgraph of K_9 (the K_6 is a subgraph of K_9 induced by the set of six end vertices). By Lemma 4.3.A, the two coloring of the K_6 subgraph contains a monochromatic triangle. If the triangle is red, then we have the desired red triangle in the K_9. If the triangle is blue, then it together with the blue edges incident with vertex v gives a blue K_4. So the claim holds if some vertex has four red edges incident to it, or has six blue edges incident to it.

ASSUME no vertex has four red edges incident to it, nor six blue edges incident to it. In K_9 every vertex is of degree eight, so every vertex must have three red edges incident and five blue edges incident to it. Then the red edges form (induce) a cubic spanning subgraph of K_9, and the blue edges form (induce) a spanning subgraph of K_9 that is regular of degree 5.

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Lemma 4.3.B (continued 2)

Proof (continued). But no graph can have an odd number of odd degree vertices by Theorem 1.1.1, a CONTRADICTION. So the assumption is false, and every vertex must have either four red edges or six blue edges incident to it. Under these conditions, the claim holds as shown above.

To show the minimality of K_9 with respect to the property, we simply need to show that K_8 can be two colored in such a way as to have neither a red K_3 nor a blue K_4. This is given in Figure 4.3.2. □

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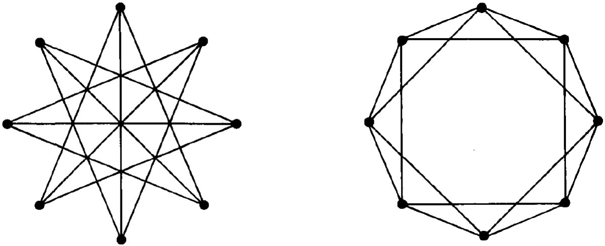

Figure 4.3.4

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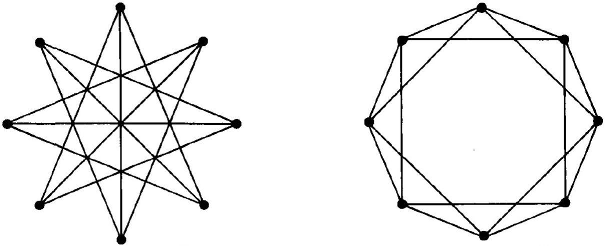

no K_3 **Figure 4.3.4** no K_4

Theorem 4.3.2

Theorem 4.3.2. For every m and n, there exists the Ramsey number $r(m, n)$ such that edge-coloring $K_{r(m,n)}$ with red and blue implies that $K_{r(m,n)}$ contains either a red K_m or a blue K_n. Furthermore, $r(m, n)$ satisfies the inequality $r(m, n) \leq r(m-1, n) + r(m, n-1)$.

Proof. We give a proof using mathematical induction on $k = m + n$. Since K_1 has no edges, the smallest case that makes sense is when $m = n = 2$ and $k = 4$. Now $r(2, 2) = 2$ by Note 4.3.A, and $r(1, 2) = r(2, 1) = 1$ by Note 4.3.A, so $r(2, 2) = 2 \leq 1 + 1 = r(1, 2) + r(2, 1)$, establishing the base case of $k = m + n = 4$.

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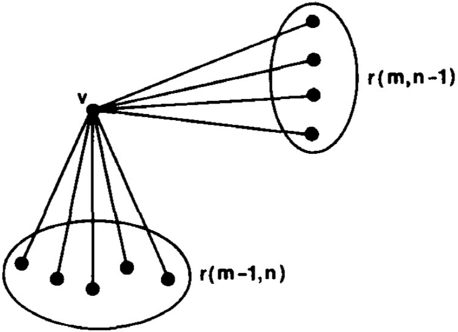
Theorem 4.3.2 (continued 1)

Proof (continued). Since G has $r(m-1, n) + r(m, n-1)$ vertices, then the degree of v is $r(m-1, n) + r(m, n-1) - 1$. So if there are strictly fewer then $r(m-1, n)$ red edges incident to v, then there must be at least $r(m, n-1)$ blue edges incident to v. That is, vertex v has either $r(m-1, n)$ red edges incident to it or $r(m, n-1)$ blue edges incident to it. We consider each of these two possibilities.



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Proof (continued). Case 1. Suppose there are $r(m-1,n)$ red edges incident to v. The subgraph H induced by the other endpoints of these edges is a complete graph with $r(m-1,n)$ vertices that is edge-colored with red and blue. Since $(m-1)+n = m+n-1 < m+n = k$, then by the induction hypothesis H contains either a red K_{m-1} or a blue K_n. Now a red K_{m-1} in H together with v and the red edges joining v to the vertices of the red K_{m-1} (these exist because v is adjacent to every vertex of H by construction) form a red K_m in G. If there is a blue K_n in H then this blue K_n is also in G (since H is a subgraph of G). That is, if there are $r(m-1,n)$ red edges at v then G either contains a red K_m or a blue K_n and the induction step is established in this case.

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Theorem 4.3.2 (continued 3)

Proof (continued). Case 2. Suppose there are $r(m, n-1)$ blue edges incident to v. The subgraph I induced by the other endpoints of these edges is a complete graph with $r(m, n-1)$ vertices that is edge-colored with red and blue. Since $m + (n-1) = m + n - 1 < m + n = k$, then by the induction hypothesis I contains either a red K_m or a blue K_{n-1}. If there is a red K_m in I then this red K_m is also in G (since H is a subgraph of G). Now a blue K_{n-1} in I together with v and the blue edges joining v to the vertices of the blue K_{n-1} (these exist because v is adjacent to every vertex of I by construction) form a blue K_n in G. That is, if there are $r(m-1, n)$ blue edges at v then G either contains a red K_m or a blue K_n and the induction step is established in this case.

Since either Case 1 or Case 2 must hold and the induction step is established in both cases, then by mathematical induction the result $r(m, n) \leq r(m-1, n) + r(m, n-1)$ for all m and n. □

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Lemma 4.3.C

Lemma 4.3.C. If the edges of $K_{5,5}$ are colored with two colors, there will be a monochromatic $K_{2,2}$.

Proof. There are 25 edges in $K_{5,5}$ so one of the colors will to at least 13 edges. Let S be the subgraph of $K_{5,5}$ induced by these edges. Since S is bipartite, then it will contain a (monochromatic) $K_{2,2}$ precisely when two vertices have two common neighbors. We consider three cases.

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Case 1. Suppose one vertex v has degree 5 in S. Since the average degree of the remaining four vertices in the partite set containing v is two (because the total degree is 13), then at lest one of these remaining vertices has degree at least two; denote it w. Since the other partite set has five vertices, then v and w must have two neighbors in common. That is, S (and hence $K_{5,5}$) contains a $K_{2,2}$.

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Proof (continued). Case 2. Suppose vertex v has degree 4 in S. Since there are at least nine edges remaining, then some vertex has degree 3 in S; denote it w. Then v and w must have two neighbors in common. That is, S (and hence $K_{5,5}$) contains a $K_{2,2}$.

Case 3. Suppose at least three vertices in one of the partite sets have degree at least 3 in S; denote them u, v, and w. At least two of u, v, w these must have two neighbors in common. That is, S (and hence $K_{5,5}$) contains a $K_{2,2}$.

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If none of the cases hold and there are no vertices of degree 4 or 5 and at most two vertices of degree 3, then we could have (at most) two vertices of degree 3 and three vertices of degree 2 for a total of (at most) 12 edges. But S has 13 edges, so these cases cover all possibilities and the claim holds. □

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