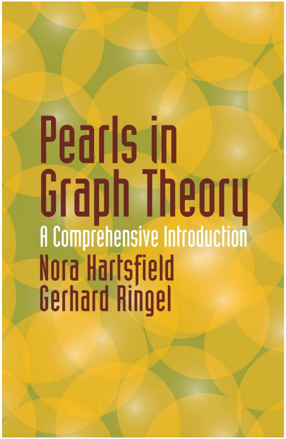
Introduction to Graph Theory

Chapter 5. Counting

5.1. Counting 1-Factors—Proofs of Theorems



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Theorem 5.1.A

Theorem 5.1.A. There are $n!$ 1-factors in $K_{n,n}$.

Proof. Color the vertices in one partite set red and the vertices in the other partite set blue. Also number the red vertices $1, 2, \ldots, n$. First red vertex 1 will be one end of some edge of the 1-factor and the other end will be a blue vertex. There are n choices for the blue vertex and so n choices for the edge containing red vertex 1. Next, red vertex 2 will be one end of some edge in the 1-factor and the other end will be a blue vertex, but not the same blue vertex that was previously used. So there are $n - 1$ choices for the blue vertex and so $n - 1$ choices for the edge containing red vertex 2.

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Theorem 5.1.A (continued)

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Proof (continued). So there are $n - (k-1) = n + 1 - k$ choices for the blue vertex and so $n + 1 - k$ choices for the edge containing red vertex k. By the Fundamental Counting Principle, the number of possible 1-factors is

$$\prod_{k=1}^{n} (n+1-k) = (n)(n-1)(n-2)\cdots(3)(2)(1) = n!,$$

as claimed. □

Note. We could also approach this with a proof based on mathematical induction.

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Theorem 5.1.B

Theorem 5.1.B. There are $\dfrac{n!}{2(n-k-1)!}$ different subgraphs of K_n isomorphic to path P_k.

Proof. Recall that P_k is a path of length k, of it has k edges and $k+1$ vertices. If $k \geq n$ then there are no such paths in K_n, so we suppose that $k < n$. Notice that every path P_k in K_n is determined by an ordered $(k+1)$-tuple of distinct vertices of K_n (but there is not a one-to-one correspondence between paths and $(k+1)$-tuples). The $(k+1)$-tuple can start at any vertex of K_n, so there are n choices for the first entry of the $(k+1)$-tuple. Next, there are $n-1$ choices for the second entry of the $(k+1)$-tuple, since the vertices must be distinct. Similarly, for the ith entry in the $(k+1)$-tuple (where $2 \leq i \leq k+1$), there are $n-(i-1)$ choices of a vertex.

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Theorem 5.1.B (continued)

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Proof (continued). By the Fundamental Counting Principle, the number of possible $(k+1)$-tuples of distinct vertices of K_n is

$$\prod{}_i = 1^{k+1}(n-(i-1)) = \prod_{i=1}^{k+1}(n+1)-i$$

$$= n(n-1)(n-2)\cdots(n-k) = \frac{n!}{(n-k-1)!}.$$

For each path P_k of K_n, there are two $(k+1)$-tuples of distinct vertices of K_n (each has the same vertices, just in opposite orders). So the number paths P_k in K_n is half the number of such $(k+1)$-tuples, namely $\dfrac{n!}{2(n-k-1)!}$ as claimed. □

Theorem 5.1.C

Theorem 5.1.C. There are $n\binom{n-1}{3}$ different subgraphs of K_n isomorphic to $K_{1,3}$.

Proof. For each subgraph of K_n isomorphic to $K_{1,3}$, we first choose the vertex of degree three and then choose the other three vertices. There are n choices for the degree three vertex and, since the degree one vertices can be any vertex except the one already chosen, there are $\binom{n-1}{3}$ choices for the degree one vertices. By the Fundamental Counting Principle, the number of subgraphs of K_n isomorphic to $K_{1,3}$ is $n\binom{n-1}{3}$, as claimed. □

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Theorem 5.1.D

Theorem 5.1.D. There are $\dfrac{(2h-1)!}{2^{h-1}(h-1)!}$ different 1-factors in K_{2h}.

Proof. Let the number of 1-factors in K_{2h} be denoted by $g(2h)$. Let x be an arbitrary vertex of K_{2h}. Choose an edge incident to x. Since x is in the complete graph K_{2h}, then its degree is $2h-1$ so that there are $2h-1$ choices for the edge incident to x. Let the other end of the chosen edge be vertex y. We now "disregard" vertices x and y and all edges incident to them, thus modifying K_{2h} to become K_{2h-2} (this process is called *edge deletion*; see my online notes for graduate Graph Theory 1 on Section 2.1. Subgraphs and Supergraphs).

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$g(2h) = (2h-1)(2h-3)g(2h-4)$.

Theorem 5.1.D (continued)

Proof. Iterating this process we have

$$\begin{aligned} g(2h) &= (2h-1)(2h-3)(2h-5)\cdots(5)(3)g(1) \\ &= (2h-1)(2h-3)(2h-5)\cdots(5)(3)(1). \end{aligned}$$

Notice that

$$\begin{aligned} 2^{h-1}(h-1)! &= (2(h-1))(2(h-2))(2(h-3))\cdots(2(3))(2(2))(2(1)) \\ &= (2h-2)(2h-4)(2h-6)\cdots(6)(4)(2). \end{aligned}$$

Therefore,

$$\begin{aligned} g(2h) &= (2h-1)(2h-3)(2h-5)\cdots(5)(3)g(1) \\ &= (2h-1)(2h-3)(2h-5)\cdots(5)(3)(1) \\ &= \frac{(2h-1)(2h-2)(2h-3)(2h-4)\cdots(4)(3)(2)(1)}{(2h-2)(2h-4)\cdots(4)(2)} \\ &= \frac{(2h-1)!}{2^{h-1}(h-1)!}, \text{ as claimed.} \qquad \square \end{aligned}$$

Theorem 5.1.E

Theorem 5.1.E. There number of different 1-factors in $K_{n,n}$ minus a 1-factor is

$$n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-1}}{(n-1)!}+\frac{(-1)^n}{n!}\right).$$

Proof. Let D_n denote the number of 1-factors in $K_{n,n}$ minus a 1-factor (that is, D_n is the number of derangements of n objects). Now $D_1 = 0$ and $D_2 = 1$. Here, we indicate the edges of the 1-factor that has been removed as dotted lines.

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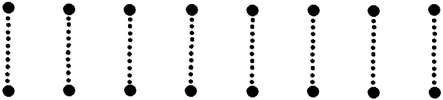

We label the vertices in one partite set as $1, 2, \ldots, n$ and those in the other partite set $1', 2', \ldots, n'$, such that the edges of the removed 1-factor are $i\,i'$ for $i \in \{1, 2, \ldots, n\}$.

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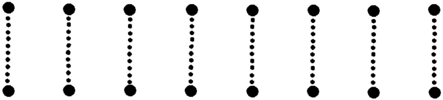

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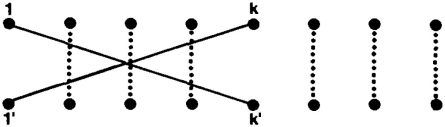
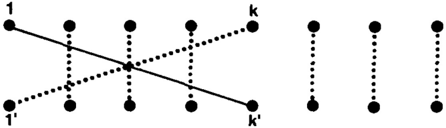
Theorem 5.1.E (continued 1)

Proof (continued). We derive an expression for D_n in terms of D_{n-1} and D_{n-2}. First, we count the number of 1-factors that contain the edges $1\,k'$ and $k\,1'$, where $n \neq 1$ and $k' \neq 1'$.



There are D_{n-2} such 1-factors for each value of $k \neq 1$ and k can assume any of the $n-1$ values $2, 3, \ldots, n$.



Next we count the number of 1-factors that contain edge $1\,k'$ but not the edge $k\,1'$. There are D_{n-1} such 1-factors for each k. Again, k can be any of the $n-1$ values $2, 3, \ldots, n$, so there are $(n-1)D_{n-1}$ such 1-factors.

Theorem 5.1.E (continued 2)

Theorem 5.1.E. There number of different 1-factors in $K_{n,n}$ minus a 1-factor is

$$n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-1}}{(n-1)!}+\frac{(-1)^n}{n!}\right).$$

Proof (continued). We now have the recurrence relation $D_n=(n-1)(D_{n-1}+D_{n-2})$. This can be rewritten as $D_n-nD_{n-1}=-(D_{n-1}-(n-1)D_{n-2})$. Replacing n with $n-1$ we then have $D_{n-1}-(n-1)D_{n-2}=-(D_{n-2}-(n-2)D_{n-3})$, which substitutes into the previous equation to give $D_n-nD_{n-1}=(-1)^2(D_{n-2}-(n-2)D_{n-3})$. Iterating this process we get $D_n-nD_{n-1}=(-1)^{n-2}(D_2-2D_1)=(-1)^n((1)-2(0))=(-1)^n$ (recall that $D_1=0$ and $D_2=1$). Rearranging gives $D_n=nD_{n-1}+(-1)^n$ or

$$\frac{D_n}{n!}=\frac{D_{n-1}}{(n-1)!}+\frac{(-1)^n}{n!}.$$

Theorem 5.1.E (continued 3)

Proof (continued). Writing out $\dfrac{D_n}{n!} = \dfrac{D_{n-1}}{(n-1)!} + \dfrac{(-1)^n}{n!}$ for $n = 2, 3, \ldots, n-1, n$ gives

$$
\begin{aligned}
\frac{D_2}{2!} &= \frac{D_1}{1!} + \frac{1}{2!} \\
\frac{D_3}{3!} &= \frac{D_2}{2!} - \frac{1}{3!} \\
\frac{D_4}{4!} &= \frac{D_3}{3!} + \frac{1}{4!} \\
&\vdots \\
\frac{D_{n-1}}{(n-1)!} &= \frac{D_{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} \\
\frac{D_n}{n!} &= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}
\end{aligned}
$$

Summing these equations and simplifying (noticing that $D_1 = 0$) gives ...

Theorem 5.1.E (continued 4)

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Proof (continued). ...

$$\frac{D_n}{n!}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-1}}{(n-1)!}+\frac{(-1)^{n}}{n!},$$

or

$$D_n=n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-1}}{(n-1)!}+\frac{(-1)^{n}}{n!}\right),$$

as claimed. □