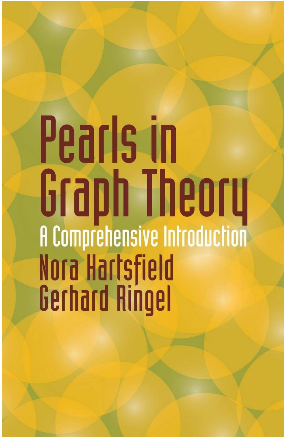
Introduction to Graph Theory

Chapter 5. Counting

5.2. Cayley's Spanning Tree Formula—Proofs of Theorems



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Proof. Based on the Fundamental Counting Principle (see my online notes for Applied Combinatorics and Problem Solving [MATH 3340] on Section 1.1. The Fundamental Counting Principle for a statement of the principle and for a list of several classes in which you might encounter it), the number of ways b_i can be chosen in n for $1 \leq i \leq n-2$, so the number of ways of choosing the sequence is n^{n-2}, as claimed. □

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Theorem 5.2.1. Cayley's Formula

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Proof. Let the vertex set of K_n be $N = \{1, 2, \ldots, n\}$. By Lemma 5.2.A, the number of sequences of length $n-2$ that can be formed from N is n^{n-2}. We will establish a one-to-one correspondence between the set of n^{n-2} sequences and the set of subtrees of K_n.

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For a given tree T, we associate a unique sequence $(t_1, t_2, \ldots, t_{n-2})$ as follows. As in the example in the notes, think of N as an ordering (and a labeling) of the vertices of the tree. Let s_1 be the first vertex of degree one in T (based on the ordering).

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Theorem 5.2.1. Cayley's Formula (continued 1)

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Proof (continued). Delete s_2 from $T - s_1$, denote by s_3 as the first vertex of degree one in $(T - s_1) - s_2$, and take the one vertex adjacent to s_3 (in tree $(T - s_1) - s_2$) as t_3. Iterate this operation until t_{n-2} has been defined and a tree with just two vertices and one edge remain. Notice that adjacency and the labeling N determine the Prüfer code, so that different spanning trees determine different sequences.

Observe that a vertex of degree 1 in tree T does not appear in the Prüfer code (but it's neighbor does appear; after the neighbor appears the degree one vertex is deleted). In general, a vertex v or tree T occurs $d(v) - 1$ (in T) times in the Prüfer code (it appears one time because of its neighbors, until is is degree one in the altered tree at which time its remaining neighbor is added to the sequence and v is deleted).

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Theorem 5.2.1. Cayley's Formula (continued 2)

Proof (continued). We now reverse the procedure, start with the Prüfer code $(t_1, t_2, \ldots, t_{n-2})$, and create a tree T from the sequence. Since v appears $d(v) - 1$ in the sequence, the vertices of tree T of degree one are precisely those that so not appear in the sequence. To create tree T from the sequence, let s_1 be the first vertex of $N = \{1, 2, \ldots, n\}$ not in $(t_1, t_2, \ldots, t_{n-1})$; join s_1 to t_1. Next, let s_2 be the first vertex in $N \setminus \{s_1\}$ which is not in $(t_2, t_3, \ldots, t_{n-2})$; join s_2 to t_2. Let s_3 be the first vertex in $N \setminus \{s_1, s_2\}$ which is not in $(t_3, t_4, \ldots, t_{n-2})$; join s_3 to t_3. Iterate this process until the $n-2$ edges $s_1 t_1$, $s_2 t_2, \ldots, s_{n-2} t_{n-2}$ result, and finally add the edge joining the two remaining vertices in $N \setminus \{s_1, s_2, \ldots, s_{n-2}\}$.

The Prüfer code and the vertex labeling N determine adjacency in this procedure, so that different sequences determine different spanning trees. Therefore different sequences determine different spanning trees. Hence there is a one-to-one correspondence between the set of n^{n-2} sequences and the set of subtrees of K_n, and the number of spanning trees in K_n is n^{n-2}, as claimed. □

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