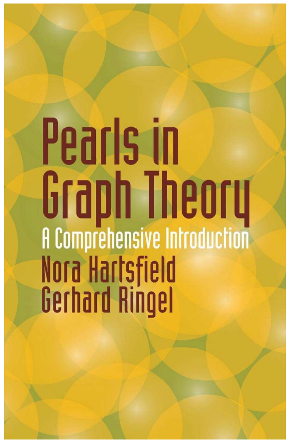
Introduction to Graph Theory

Chapter 5. Counting

5.3. More Spanning Trees—Proofs of Theorems



Theorem 5.3.1

Theorem 5.3.1. The number of spanning trees in the graph $K_{2,n}$ is $n2^{n-1}$.

Proof. We treat the graph $K_{2,n}$ as if it has two blue vertices, a and b, and n red vertices; see Figure 5.3.1 below. In any spanning tree of $K_{2,n}$, there is a unique path between a and b by Theorem 1.3.5, and we see that this path must be length two with center vertex, say, x. We now apply the Fundamental Counting Principle (see my online notes for Applied Combinatorics and Problem Solving [MATH 3340] on Section 1.1. The Fundamental Counting Principle. First, we choose the center vertex x (this determines the unique path joining a and b in a spanning tree); there are n choices for this. Second, for each of the remaining $n - 1$ vertices in the partite set of size n, we either join the vertex to a or to b there are 2^{n-1} choices for this. Therefore, there are $n2^{n-1}$ choices for a spanning tree of $K_{2,n}$, as claimed.

Theorem 5.3.1 (continued)

Theorem 5.3.1. The number of spanning trees in the graph $K_{2,n}$ is $n2^{n-1}$.

Proof (continued).

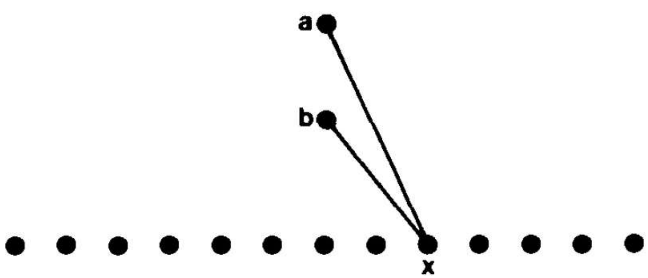


Figure 5.3.1

□

Theorem 5.3.2

Theorem 5.3.2. The number of spanning trees in the graph $K_{3,n}$ is $n^2 3^{n-1}$.

Proof. We treat the graph $K_{3,n}$ as if it has three blue vertices, a, b, and c, and n red vertices. In any spanning tree of $K_{3,n}$ either each of the pairs a, b, a, c, and b, c has distance two as in Figure 5.3.2, or one of the pairs has distance four as in Figure 5.3.3.

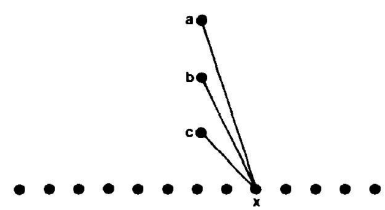


Figure 5.3.2

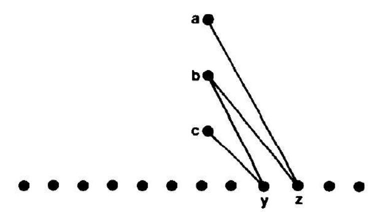


Figure 5.3.3

Theorem 5.3.2 (continued)

Proof (continued). We count the number of spanning trees of the form as given in Figure 5.3.2. First, there are n ways to choose vertex x. Second, for each of the remaining $n-1$ red vertices, there are three possible edges in the spanning tree; either the vertex is adjacent to a, b, or c. So by the Fundamental Counting Principle, there are $n3^{n-1}$ such spanning trees.

Now we count the number of spanning trees of the form given in Figure 5.3.3. If the red vertices y and z are chosen, then there are six possibilities for the path of length four, since there are three ways to choose the center point (labeled b in Figure 5.3.3) and y and z can be interchanged. There are $\binom{n}{2} = n(n-1)/2$ ways to choose the pair y and z. For each of the remaining $n-2$ red vertices there are three possible edges in the spanning tree; either the vertex is adjacent to a, b, or c. So by the Fundamental Counting Principle, there are $n3^{n-1} + n(n-1)3^{n-1} = n^2 3^{n-1}$ such spanning trees. Therefore the total number of spanning trees of $K_{3,n}$ is $n3^{n-1} + n(n-1)3^{n-1} = n^2 3^{n-1}$, as claimed. □