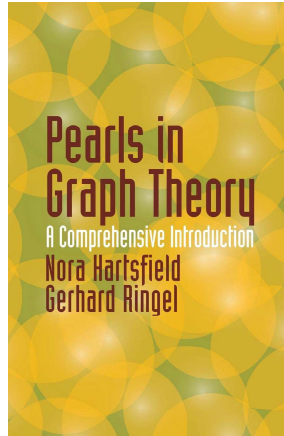


Introduction to Graph Theory

Chapter 6. Labeling Graphs

6.1. Magic Graphs and Graceful Trees—Proofs of Theorems



Theorem 6.1.2

Theorem 6.1.2. If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.

Proof. Since G is bipartite then the length of the Hamilton cycle is even, say $2n$ (notice that this implies that each partite set must be size n). The number of edges in G is then $q = 2(2n) = 4n$. Choose an arbitrary vertex a and label the edges of the first Hamilton cycle starting at a by $4n - 1, 1, 4n - 3, 3, 4n - 5, 5, \dots, 4n - (2k - 1), 2k - 1, \dots, 2n + 1, 2n - 1$ (all odd numbers; notice that $1 \leq k \leq n$). Then label the edges of the second Hamilton cycle, starting at a , by $2, 4n, 4, 4n - 2, 6, 4n - 4, \dots, 2k, 4n + 2 - 2k, \dots, 2n, 2n + 2$ (all even numbers; notice that $1 \leq k \leq n$). This is illustrated for $n = 5$ in Figure 6.1.7 of the notes.

Theorem 6.1.2 (continued)

Theorem 6.1.2. If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.

Proof (continued). Since G is bipartite, the vertices can be colored red and blue with no two adjacent vertices of the same color. If a is blue, then the sum of the odd-numbered edges at all the blue vertices except a is $4n - (2k - 1) + (2k - 1) = 4n - 2$. The sum of the even-numbered edges at all blue vertices except a is $(4n + 2 - 2k) + (2(k + 1)) = 4n + 4$. The sum of all edges at a is $(4n - 1) + (2n - 1) + (2) + (2n + 2) = 8n + 2$. Therefore the sum of all edges at any blue vertex is $(4n - 2) + (4n + 4) = 8n + 2$.

The sum of the odd-numbered edges at each red vertex is $(4n - (2k - 1)) + (2k - 1) = 4n$. The sum of the even-numbered edges at each red vertex is $(2k) + (4n + 2 - 2k) = 4n + 2$. Hence the sum of all edges at any red vertex is $8n + 2$. Therefore, the sum of all edges incident to a any vertex of G is $8n + 2$ and G is magic, as claimed. \square

Theorem 6.1.3

Theorem 6.1.3. If a graph G is decomposable into two magic spanning subgraphs G_1 and G_2 where G_2 is regular, then G is magic.

Proof. Let q_1 and q_2 denote the number of edges of G_1 and G_2 , respectively. Consider a magic labeling of G_1 (so the edge labels are $1, 2, \dots, q_1$) and a magic labeling of G_2 (where the edge labels are $1, 2, \dots, q_2$). To each label of G_2 , add q_2 . Since G_2 is regular, we have added the same amount at each vertex. We now have the edges of G labeled with the $1, 2, \dots, q_1, q_1 + 1, \dots, q_1 + q_2$, and the sum of the labels at each vertex is the same (namely, the sum at every vertex is what it is in G_1 plus what it is in G_2 plus kq_2 where k is the degree of each vertex of regular graph G_2). That is, G is magic, as claimed. \square