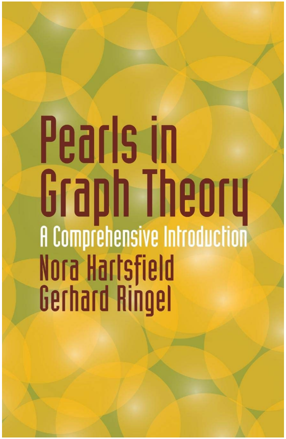
Introduction to Graph Theory

Chapter 6. Labeling Graphs

6.1. Magic Graphs and Graceful Trees—Proofs of Theorems



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Theorem 6.1.2

Theorem 6.1.2. If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.

Proof. Since G is bipartite then the length of the Hamilton cycle if even, say $2n$ (notice that this implies that each partite set must be size n). The number of edges in G is then $q = 2(2n) = 4n$. Choose an arbitrary vertex a and label the edges of the first Hamilton cycle starting at a by $4n-1, 1, 4n-3, 3, 4n-5, 5, \ldots, 4n-(2k-1), 2k-1, \ldots 2n+1, 2n-1$ (all odd numbers; notice that $1 \leq k \leq n$). Then label the edges of the second Hamilton cycle, starting at a, by $2, 4n, 4, 4n-2, 6, 4n-4, \ldots, 2k, 4n+2-2k, \ldots, 2n, 2n+2$ (all even numbers; notice that $1 \leq k \leq n$). This is illustrated for $n = 5$ in Figure 6.1.7 of the notes.

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Theorem 6.1.2 (continued)

Theorem 6.1.2. If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.

Proof (continued). Since G is bipartite, the vertices can be colored red and blue with no two adjacent vertices of the same color. If a is blue, then the sum of the odd-numbered edges at all the blue vertices except a is $4n - (2k-1) + (2k-1) = 4n - 2$. The sum of the even-numbered edges at all blue vertices except a is $(4n + 2 - 2k) + (2(k+1)) = 4n + 4$. The sum of all edges at a is $(4n-1) + (2n-1) + (2) + (2n+2) = 8n + 2$. Therefore the sum of all edges at any blue vertex if $(4n-2) + (4n+4) = 8n + 2$.

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The sum of the odd-numbered edges at each red vertex is $(4n - (2k-1)) + (2k-1) = 4n$. The sum of the even-numbered edges at each red vertex is $(2k) + (4n + 2 - 2k)) = 4n + 2$. Hence the sum of all edges at any red vertex is $8n + 2$. Therefore, the sum of all edges incident to a any vertex of G is $8n + 2$ and G is magic, as claimed. □

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The sum of the odd-numbered edges at each red vertex is $(4n-(2k-1))+(2k-1)=4n$. The sum of the even-numbered edges at each red vertex is $(2k)+(4n+2-2k))=4n+2$. Hence the sum of all edges at any red vertex is $8n+2$. Therefore, the sum of all edges incident to a any vertex of G is $8n+2$ and G is magic, as claimed. □

Theorem 6.1.3

Theorem 6.1.3. If a graph G is decomposable into two magic spanning subgraphs G_1 and G_2 where G_2 is regular, then G is magic.

Proof. Let q_1 and q_2 denote the number of edges of G_1 and G_2, respectively. Consider a magic labeling of G_1 (so the edge labels are $1, 2, \ldots, q_1$) and a magic labeling of G_2 (where the edge labels are $1, 2, \ldots, q_2$). To each label of G_2, add q_2. Since G_2 is regular, we have added the same amount at each vertex. We now have the edges of G labeled with the $1, 2, \ldots, q_1, q_1 + 1, \ldots, q_1 + q_2$, and the sum of the labels at each vertex is the same (namely, the sum at every vertex is what it is in G_1 plus what it is in G_2 plus kq_1 where k is the degree of each vertex of regular graph G_2). That is, G is magic, as claimed. □

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