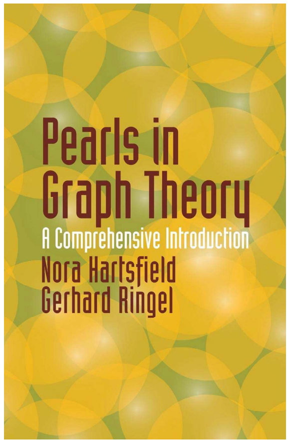
Introduction to Graph Theory

Chapter 6. Labeling Graphs

6.2. Conservative Graphs—Proofs of Theorems



Theorem 6.2.1

Theorem 6.2.1. If graph G is decomposable into two Hamilton cycles, then G is conservative.

Proof. Let G have n vertices. Choose a vertex a and, starting at a, traverse the edges of one Hamilton cycle, orienting the edges in the same directions as you go (turning the edges into arcs and the graph into a directed graph). Label the arcs beginning at a by $1, 3, 5, \ldots, 2n-1$ (all odd numbers). Then, stating at a, traverse the edges of the other Hamilton cycle, orienting the edges in the same direction as you go, and labeling the arcs beginning at a by $2n, 2n-2, 2n-4, \ldots, 4, 2$ (all even numbers). At each vertex of the first Hamilton cycle (except a) there is a net increase of two in the arc labels, and at each vertex of the second Hamilton cycle (except a) there is a net decrease of two. So the condition of being a conservative graph is satisfied at vertices, except possibly a.

Theorem 6.2.1 (continued)

Theorem 6.2.1. If graph G is decomposable into two Hamilton cycles, then G is conservative.

Proof (continued). At vertex a there are in-arcs with labels 2 and $2n-1$ (summing to $2n+1$), and there are out-arcs with labels $2n$ and 1 (summing to $2n+1$) so that the condition of being a conservative graph is satisfied at vertex a also. Therefore, Kirchhoff's Current Law holds at every vertex of G. That is, G is a conservative graph, as claimed. □

Theorem 6.2.2. Kirchhoff's Global Current Law

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If G is a labeled, directed graph such that Kirchhoff's Current Law at every vertex of G except a particular vertex a, then Kirchhoff's Current Law also holds at the vertex a.

Proof. Suppose that the degree of a is $h+f$, where the incoming arcs at a are labeled $c_1, c_2, \ldots, c_k$, the outgoing arcs at a are labeled $b_1, b_2, \ldots, b_f$, and the other arcs in the directed graph are labeled with d_i. Let S be the set of all vertices except a. By hypothesis, Kirchoff's Current Law holds at every vertex in S. So at each vertex in S, if we add the labels on the in-arcs and add the labels on the out-arcs, the directed sum will be zero. Each label on an arc is counted exactly once as a label on an in-arc (at the head of the arc containing the label) and is counted exactly once as a label on an out-arc (at the tail of the arc containing the label).

Theorem 6.2.2. Kirchhoff's Global Current Law (continued)

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If G is a labeled, directed graph such that Kirchhoff's Current Law at every vertex of G except a particular vertex a, then Kirchhoff's Current Law also holds at the vertex a.

Proof (continued). So, summing over all vertices of G, we have that the sum of the in-arc label minus the sum of the out-arc labels is 0:

$$(d_1 + d_2 + \cdots + d_\ell) + (b_1 + b_2 + \cdots + b_k)$$

$$-(d_1 + d_2 + \cdots + d_\ell) - (c_1 - c_2 - \cdots - c_k) = 0.$$

Rearranging we have $b_1 + b_2 + \cdots + b_f = c_1 + c_2 + \cdots + c_k$, so that Kirchhoff's Current Law holds at a, as claimed. □

Theorem 6.2.3

Theorem 6.2.3. If G is decomposable into two subgraphs H_1 and H_2, and if H_1 is conservative, and H_2 is strongly conservative, then G is conservative. Moreover, if both H_1 and H_2 are strongly conservative, then G is strongly conservative.

Proof. Let q_1 be the number of edges in H_1 and q_2 the number of edges in H_2, so that the number of edges in G is $q_1 + q_2$. Let $1, 2, 3, \ldots, q_1$ be a conservative labeling of H_1, and let $q_1 + 1, q_1 + 2, \ldots, q_1 + q_2$ be a strongly conservative labeling of H_2. In both of these labelings Kirchhoff's Current Law holds, so the directed sum at each vertex of G is zero. The $q_1 + q_2$ arcs (i.e., oriented edges) of G are then labeled $1, 2, 3, \ldots, q_1, q_1 + 1, q_1 + 2, \ldots, q_1 + q_2$ and Kirchhoff's Current Law is satisfied at each vertex, so that G is conservative, as claimed.

Theorem 6.2.3 (continued)

Theorem 6.2.3. If G is decomposable into two subgraphs H_1 and H_2, and if H_1 is conservative, and H_2 is strongly conservative, then G is conservative. Moreover, if both H_1 and H_2 are strongly conservative, then G is strongly conservative.

Proof (continued). Now suppose that both H_1 and H_2 are strongly conservative, and let h be any given number. Let $h + 1, h + 2, \ldots, h + q_1$ be a strongly conservative labeling of H_1, and let $h + q_1 + 1, h + q_1 + 2, \ldots, h + q_1 + q_2$ be a strongly conservative labeling of H_2. Again Kirchhoff's Current Law is satisfied at each vertex by both labelings, so the directed sum at each vertex in G is zero. The edges are labeled $h + 1, h + 2, \ldots, h + q_1, h + q_1, \ldots, h + q_1 + q_2$, and hence G is strongly conservative, as claimed. □

Theorem 6.2.4

Theorem 6.2.4. If G is a graph with n vertices, where n is odd, and G is decomposable into three Hamilton cycles, then G is strongly conservative.

Proof. Denote the three Hamilton cycles as H_1, H_2, and H_3. Choose a vertex a and, starting at a, traverse the edges of H_3, orienting the edges in the same directions as you go (turning the edges into arcs and the graph into a directed graph). Label the arcs beginning at a by $3n, 3n - 6, 3n - 12, \ldots, 15, 9, 3, 3n - 3, 3n - 9, \ldots, 18, 12, 6$ (this is n labels, all of which are 0 modulo 3). Denote by b the vertex between the arc labeled 3 and the arc labeled $3n - 3$ in H_3. Next, starting at a traverse the edges of H_1, orienting the edges in the same directions as you go (turning the edges into arcs and the graph into a directed graph), and labeling the arcs beginning at a by $1, 4, 7, \ldots, 3n - 5, 3n - 2$ (this is n labels, all of which are 1 modulo 3).

Theorem 6.2.4 (continued 1)

Proof (continued). Finally, for H_2 start at vertex b defined above, traverse the edges of H_2, orienting the edges in the same directions as you go (turning the edges into arcs and the graph into a directed graph), and labeling the arcs beginning at b by $2, 5, 8, \ldots, 3n-4, 3n-1$ (this is n labels, all of which are 2 modulo 3).

For every vertex of G, except for a and b, there is a net decrease of six in H_3, a net increase of three in H_1, and a net increase of three in H_2. So Kirchhoff's Current Law holds at every vertex, except possibly vertices a and b. At vertex a, the in-arcs have labels 6, $3n-2$, and t for some $t \equiv 2 \pmod 3$ (in H_3, H_1, and H_2, respectively). The out-arcs at a have labels $3n$, 1, and $t+3$ (in H_3, H_1, and H_2, respectively; notice that there is a net increase of three in H_2 except at vertex $b \neq a$ so that whatever the in-arc label at a in H_2 is, say t, the out-arc in H_2 is $t+3$). So the directed sum is $(6+(3n-2)+t)-(3n+1+(t+3))=0$ and Kirchhoff's Current Law holds at a.

Theorem 6.2.4 (continued 2)

Theorem 6.2.4. If G is a graph with n vertices, where n is odd, and G is decomposable into three Hamilton cycles, then G is strongly conservative.

Proof (continued). By Theorem 6.2.2, Kirchoff's Current Law must hold at vertex b. Since at each vertex of the orientation of G there are three in-arcs and three out-arcs, then we can add h to each of the labels above and Kirchhoff's Current Law will still hold. That is, G is strongly conservative, as claimed. □

Theorem 6.2.5

Theorem 6.2.5. If n is odd, $n \geq 5$, then K_n is conservative.

Proof. With n odd, K_n is decomposable into $(n-1)/2$ Hamilton cycles by Theorem 2.3.1. By Theorem 6.2.1*, the union of two Hamilton cycles is strongly conservative, and by Theorem 6.2.3 if G is decomposable into two strongly conservative graphs, then G is strongly conservative. So if $(n-1)/2$ is even, then by repeated application of Theorem 6.2.3 (or by induction) we have that K_n is strongly conservative. That is, if $n \equiv 1 \pmod 4$, $n \geq 5$, then K_n is strongly conservative (and hence conservative), as claimed. By Theorem 6.2.4, the union of three Hamilton cycles is strongly conservative. So if $(n-1)/2$ is odd (and so can be written as a sum of the form $2+2+2+\cdots+2+3$), then by repeated application of Theorem 6.2.3 (or by induction) we have that K_n is strongly conservative. That is, if $n \equiv 3 \pmod 4), n \geq 5$, then K_n is strongly conservative (and hence conservative), as claimed. □

Theorem 6.2.6

Theorem 6.2.6. For $n \geq 3$, the wheel with n spokes, W_n, is conservative.

Proof. For n odd, direct the outer cycle in a "clockwise" direction, as given in Figure 6.2.10. Label the arcs in the directed cycle as shown, using labels $2; 3, 5, 7, \ldots, n-4, n-2; n+2, n+4, \ldots, 2n-3, 2n-1; 2n$ (a total of n labels). Direct the arcs that are spokes alternating from in-arcs to out-arcs from the center of the wheel, as given in Figure 6.2.10 (notice that the consecutive arcs labeled 1 and $2n-4$ are both out-arcs from the center, a necessity since n is odd). Label the arcs that are spokes as given in Figure 6.2.10 using the labels $1; 4, 6, 8, \ldots, 2n-6, 2n-4; n; 2n-2$ (a total of n labels).

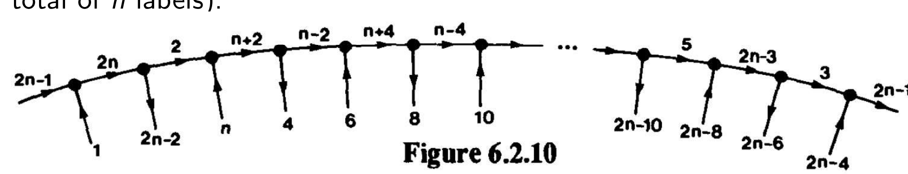

Figure 6.2.10

Theorem 6.2.6 (continued 1)

Proof (continued). We see that every number from 1 to $2n$ appears as an arc label exactly once. It is straightforward (bit a little tedious) to check that Kirchhoff's Current Law holds at all vertices of the directed cycle. Therefore, by Theorem 6.2.2, Kirchhoff's Current Law also holds at the center vertex, and G is conservative for n odd, as claimed.

For n even, direct the outer cycle in a "clockwise" direction except for one arc, as given in Figure 6.2.11 below. Label the arcs in the directed cycle as shown, using labels $2; 3, 5, 7, \ldots, n-3, n-1; n+3, n+5, \ldots, 2n-3, 2n-1; 2n-2$ (a total of n labels). Direct the arcs that are spokes alternating from in-arcs to out-arcs from the center of the wheel, as given in Figure 6.2.11 (notice that the consecutive arcs labeled 1 and $2n$ are both in-arcs from the center, and the consecutive arcs labeled $n+1$ and 4 are both out-arcs from the center). Label the arcs that are spokes as given in Figure 6.2.11 using the labels $1; 4, 6, 8, \ldots, 2n-6, 2n-4; n+1; 2n$ (a total of n labels).

Theorem 6.2.6 (continued 2)

Theorem 6.2.6. For $n \geq 3$, the wheel with n spokes, W_n, is conservative.

Proof (continued).

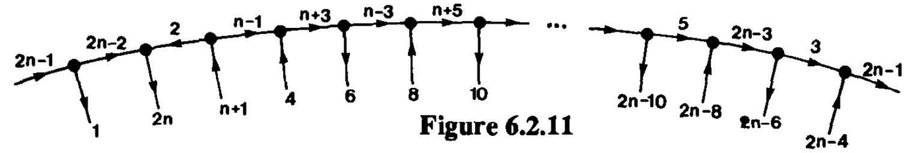

Figure 6.2.11

We see that every number from 1 to $2n$ appears as an arc label exactly once. It is straightforward (bit a little tedious) to check that Kirchhoff's Current Law holds at all vertices of the directed cycle. Therefore, by Theorem 6.2.2, Kirchhoff's Current Law also holds at the center vertex, and G is conservative for n even, as claimed. Hence, W_n satisfies Kirchhoff's Current Law at every vertex for all $n \geq 3$, so that such W_n is conservative, as claimed. □

Theorem 6.2.7

Theorem 6.2.7. If n is even, $n \geq 4$, then K_n is conservative.

Proof. Notice that K_n is the disjoint edge union of W_{n-1} and $K_{n-1} - C_{n-1}$; the graph $K_{n-1} - C_{n-1}$ is the complete graph K_{n-1} with the edges of a Hamilton cycle of K_{n-1}, and $W n - 1$ consists of the "missing" cycle C_{n-1}, the extra nth vertex, and all edges between the C_{n-1} and the $n$the vertex. The wheel W_{n-1} is conservative by Theorem 6.2.6 (this is where $n \geq 4$ is needed). Since $n-1$ is odd, then K_{n-1} is decomposable into Hamilton cycles by Theorem 2.3.1 (one of which we take to the the C_{n-1} referenced above). As in the proof of Theorem 6.2.5, we now have that $K_{n-1} - C_{n-1}$ (for $n \neq 6$) is strongly conservative by Theorems 6.2.1*, 6.2.3, and 6.2.4. Therefore, by Theorem 6.2.3, K_n is conservative for n even, $n \geq 4$, $n \neq 6$, as claimed.

Theorem 6.2.7 (continued)

Theorem 6.2.7. If n is even, $n \geq 4$, then K_n is conservative.

Proof (continued). We still need to show that K_6 is conservative. We do so with a specific labeling:

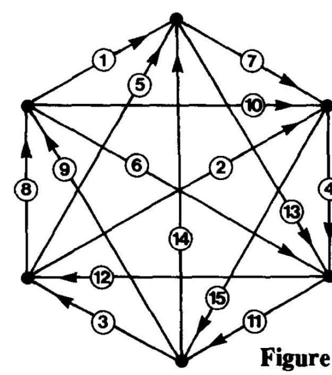

Figure 6.2.12

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