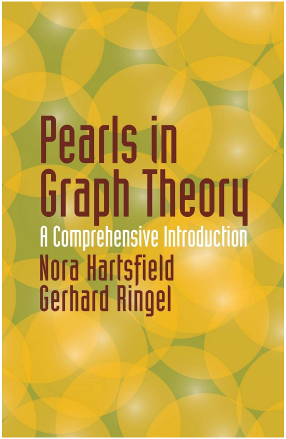
Introduction to Graph Theory

Chapter 8. Drawings of Graphs

8.1. Planar Graphs—Proofs of Theorems



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If a plane drawing of a connected graph with p vertices and q edges has r regions, then $p - q + r = 2$.

Proof. We give an induction argument on the number of cycles in the graph. If connected graph G has no cycles, then G is a tree and, by Theorem 1.3.2, $p = q + 1$, and in a plane drawing of G there is $r = 1$ region. So $p - q + r = (q + 1) - q + (1) = 2$, and the base case is established.

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For the induction hypothesis, suppose the formula holds for all plane drawings of connected graphs with fewer than n cycles. Given a plane drawing of a connected graph G with n cycles, p vertices, q edges, and r regions, consider one cycle C of G. By the Jordan Curve Theorem, C divides the plan into an inside and an outside. Let e be an edge of C. Edge e is on the boundary of two distinct regions; one of these regions is inside C and the other is outside C.

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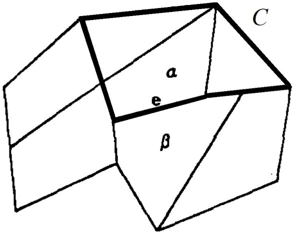
Theorem 8.1.1. Euler's Polyhedral Formula (continued)

Proof (continued).



Denote the region inside C as α and the region outside C as β. If we remove edge e, then regions α and β merge into one region. The plane drawing of $G - e$ has p vertices, $q - 1$ edges, and $r - 1$ regions. Since $G - e$ has fewer than c cycles, then by the induction hypothesis we have $p - (q - 1) + (r - 1) = 2$. This simplifies to $p - q + r = 2$, establishing the induction step. Therefore, by Mathematical Induction, the formula holds. □

Theorem 8.1.2

Theorem 8.1.2. If G is a maximal planar graph with p vertices and q edges, where $p \geq 3$, then $q = 3p - 6$.

Proof. Consider a plane drawing of G with r regions. Since G is a maximal planar graph, then by Note 8.1.B each region is bounded by three edges. In addition, every edge lies on two regions. We now count the number h of pairs (t, e) where t is a triangle, and e is an edge of t. Since each triangle contains three edges, and there are r triangles, then we have r choices for t and for each such choice we have 3 choices for e. Hence $h = 3r$.

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Theorem 8.1.4

Theorem 8.1.4. The graph K_5 is not planar.

Proof. ASSUME K_5 is planar. By Theorem 8.1.3, we must have $q \leq 3p - 6$. But for K_5, for which $p = 5$ and $q = 10$, this implies that $(10) \leq 3(5) - 6 = 9$, a CONTRADICTION. So the assumption that K_5 is planar is false, and hence K_5 is not planar, as claimed. □

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Theorem 8.1.5

Theorem 8.1.5. If G is a planar bipartite graph with p vertices and q edges, where $p \geq 3$, then $q \leq 2p - 4$.

Proof. Consider a plane drawing of G with r regions. Since G is bipartite, then by Theorem 2.1.2 G contains no odd length cycles, so that each region is bounded by [at least] four edges. In addition, every edge lies on two regions. We now count the number h of pairs (s, e) where s is a region, and e is an edge of s. Since each region contains at least four edges, and there are r regions, then we have r choices for s and for each such choice we have a least 4 choices for e. Hence $h \geq 4r$.

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Theorem 8.1.7

Theorem 8.1.7. Every planar graph contains at least one vertex of degree at most 5.

Proof. ASSUME not and that G is some planar graph with all vertices of degree at least 6. Then in G, by Theorem 1.1.1 we have $2q = \sum_{v \in V} deg(v) \geq \sum_{v \in V} 6 = 6p$, of $q \geq 3p$. But then Theorem 8.1.3 requires that a planar graph satisfies $q \leq 3p - 6$, a CONTRADICTION. So the assumption that such a graph exists is false, and the claim holds. □

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Theorem 8.1.8

Theorem 8.1.8. Suppose G s a maximal planar graph with p vertices and q edges, where $p \geq 4$. Let p_i denote the number of vertices of degree i. Then

$$3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + 3p_9 + 4p_{10} + \cdots .$$

Proof. Notice that p is the sum of the p_i's, and $2q$ is the sum of the degrees of G by Theorem 1.1.1. That is,

$$p = \sum_{i=3}^{\infty} p_i \text{ and } 2q = \sum_{i=3}^{\infty} ip_i$$

(these are "formal series," since after some point all of the p_i's are 0 and the sum is in fact finite). Since G is a maximal planar graph, then by Theorem 8.1.2, $q = 3p - 6$ or $6p - 2q = 12$. So if we multiply the first equation by six and subtract the second equation, we get...

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Proof (continued). ...So if we multiply the first equation by six and subtract the second equation, we get

$$6p - 2q = 6\sum_{i=3}^{\infty} p_i - \sum_{i=3}^{\infty} ip_i = \sum_{i=3}^{\infty}(6-i)p_i = 12,$$

or

$$3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + 3p_9 + 4P_{10} + \cdots + (k-6)P_k + \cdots ,$$

as claimed. □