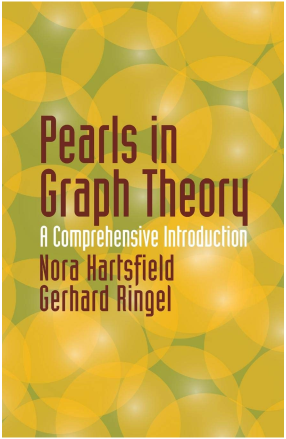
Introduction to Graph Theory

Chapter 8. Drawings of Graphs

8.2. The Four Color Theorem—Proofs of Theorems



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Theorem 8.2.1

Theorem 8.2.1. If a normal map has a coloring of the countries with four colors, then the edges of the map can be properly colored by three colors.

Proof. Denote the four colors of the map coloring as 1, 2, 3, 4. We will color the edges with colors a, b, c. Since an edge lies between two countries, this gives $\binom{4}{2} = 6$ possible pairs of colors for the two countries on the sides of an edge. We assign colors to the edges with the following "recipe."

If an edge lies between countries colored	color the edge
1 and 2	a
3 and 4	a
1 and 3	b
2 and 4	b
1 and 4	c
2 and 3	c

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2 and 3	c

Theorem 8.2.1 (continued)

Theorem 8.2.1. If a normal map has a coloring of the countries with four colors, then the edges of the map can be properly colored by three colors.

Proof (continued). Figure 8.2.3 shows the recipe, and Figure 8.2.4 gives all possible ways that three countries in a normal map colored with four colors can meet at a vertex (modulo rotations and reflections).

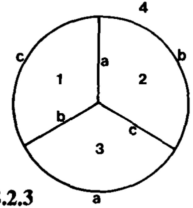

Figure 8.2.3

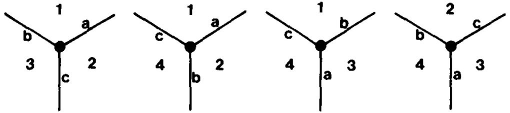

Figure 8.2.4

In each case, we have a proper edge coloring. □

Lemma 8.2.2

Lemma 8.2.2. Given a finite number of simple closed curves (cycles) in the plane that only intersect at points (not along segments of the curves), the regions defined by these curves are colorable by two colors.

Proof. The proof is based on the Jordan Curve Theorem (though Hartsfield and Ringel do not explicitly mention this). Notice that the curves may or may not intersect and the result will still hold; see Figures 8.2.5 and 8.2.6.

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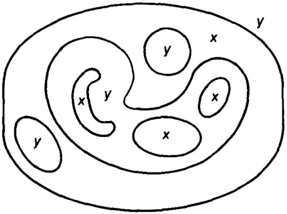
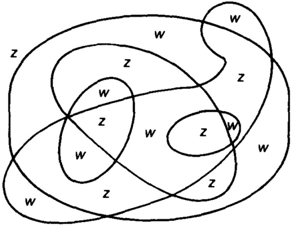


Figure 8.2.5



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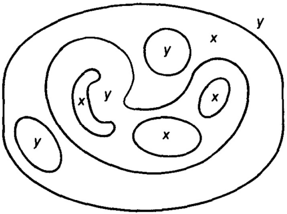

Figure 8.2.5

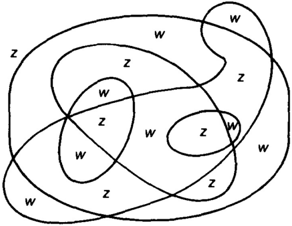

Figure 8.2.6

Lemma 8.2.2 (continued)

Lemma 8.2.2. Given a finite number of simple closed curves (cycles) in the plane that only intersect at points (not along segments of the curves), the regions defined by these curves are colorable by two colors.

Proof (continued). By the Jordan Curve Theorem, each cycle divides the plane into an inside and an outside. If a region is inside an even number of curves, color it x. If a region is inside an odd number of curves, color is y. Since the boundary between any two of these regions is part of one of the simple closed curves, one of the regions will be inside exactly one more curve than the other (this is why we must impose the "only intersect at points" condition). Hence one will be inside an even number of curves and the other will be inside an odd number of curves, so that they have different colors, as claimed. □

Theorem 8.2.3

Theorem 8.2.3. If the edges of a normal map can be properly colored by three colors, then the countries of the map can be colored by four colors.

Proof. Let M be a normal map which is properly edge colored with the three colors a, b, c. First, consider the (induced) subgraph of M that consists of all the edges labeled a or b. The subgraph consists of a set of cycles that only intersect at vertices of the subgraph. By Lemma 8.2.2, the regions defined by these cycles are colorable by two colors, say x and y. In this coloring, every country of original map M receives either color x or color y (but countries adjacent in M can get the same color; namely, those separated by an edge of color c will get the same color under this scheme).

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Proof (continued). Now each country of M has received one of the four pairs xz, xw, yz, or yw. We consider these four pairs as four colors. Consider two adjacent countries in M. If the edge between these countries is labeled b, then an x-color (either xz or xw) and the other has a y-color (either yz or yw), so their colors are different. If the edge between these countries is labeled c, then an z-color (either xz or yz) and the other has a w-color (either xw or yw), so their colors are different. If the edge between these countries is labeled a, then an x-color (either xz or xw) and the other has a y-color (either yz or yw) so their colors are different; this case can also be argued in terms of z-colors and w-colors. Hence, M is colorable with four colors, as claimed. □

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Theorem 8.2.4. If in a normal map the edges are properly colored by three colors, then the vertices can be labeled black and white so that around any given country, the number of black vertices minus the number of white vertices is always a multiple of three.

Proof. In a proper edge coloring of a normal map (which is, by definition, cubic), at each vertex all three colors appear exactly once. If the edges have colors a, b, c occurring clockwise around a vertex, color it black. If they occur counterclockwise around a vertex, then color the vertex white. See Figure 8.2.7.

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Figure 8.2.7

Theorem 8.2.4 (continued 1)

Proof (continued). We construct a new map, M', from M by replacing each white vertex of M by a triangle with black vertices, as in Figure 8.2.8.

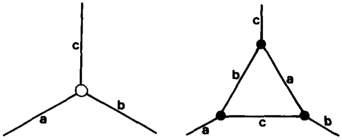

Figure 8.2.8

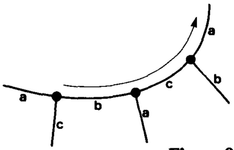

Figure 8.2.9

Notice that the new b colored edge is "between" an a colored edge and a c colored edge (and similar for the new a and c colored edges; this is where the clockwise/counterclockwise black/white coloring is needed). The edges and vertices around a given country A of map M' (which results from a counterclockwise colored region of map M) are as in Figure 8.2.9. Notice that the number of black vertices around A is a multiple of three, since going counterclockwise around the inside of A, an a is always followed by a b, a b is always followed by a c, and a c is always followed by an a.

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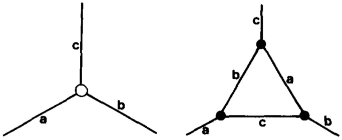

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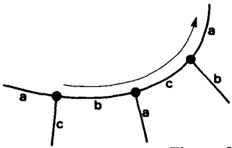

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Theorem 8.2.4 (continued 2)

Proof (continued). So in map M', all the vertices are black and the number of vertices around every country is a multiple of three. We return to the original map M by restoring the white vertices. Every time a white vertex is restored, we shrink a triangle resulting on one more white vertex and two fewer black vertices around a given country. We started with a multiple of three black vertices in the country of M' (say $3K$), and have reintroduced all white vertices of M (say W white vertices are returned) by effectively turning two black vertices of the country in M' into a single white vertex in the corresponding country in M. We started with $3K$ black vertices, picked up W white vertices, and lost $2W$ black vertices so that the resulting number of black vertices is $B = 3K - 2W$ and the number of white vertices is W. Therefore, in a region of the original map M, the number of black vertices minus the number of white vertices is $B - W = (3K - 2W) - W = 3K - 3W = 3(K - W)$, which is a multiple of three, as claimed. □

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Proof. Let M be a normal map with a labeling of the vertices by black and white with the stated property. We will construct a normal map M' by blowing up every white vertex into a triangle with three black vertices, similar to the proof of Theorem 8.2.4. For a given country of M, with the number of black vertices of the country as B and the number of white vertices of the country as W, we have by hypothesis that $B - W = 3K$ for some $k \in \mathbb{Z}$.

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Theorem 8.2.5 (continued 1)

Proof (continued). We start at any country and walk counterclockwise around the inside of the country and color the edges a, b, c, a, b, c, ... as we come to them. The number of edges around each country is a multiple of three, so the coloring is proper (so far). Next, we color a region adjoining the first region by the same scheme (since at least one edge will already be colored, the coloring is determined; since all cycles are of lengths multiples of three, if there are multiple edges already colored they will be consistent with the scheme). We use Figure 8.2.10 to justify that the resulting edge coloring of M' is proper.

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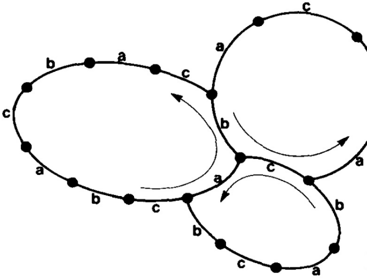

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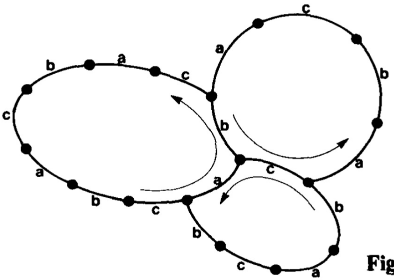

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Proof (continued). As in the proof of Theorem 8.2.4, we shrink the triangles with three black vertices down to a single white vertex and keep the coloring of the remaining edges. This gives a proper edge coloring of M, since we shrink a triangle with edges colored a, b, and c and the edges remaining around the white vertex are then colored a, b, c. The black vertices of M keep the same edges incident to them and so inherit the proper coloring from M'.