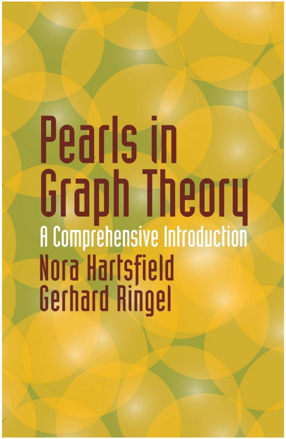
Introduction to Graph Theory

Chapter 8. Drawings of Graphs

8.3. The Five Color Theorem—Proofs of Theorems



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Lemma 8.3.1

Lemma 8.3.1. No map in the plane has five mutually adjacent countries.

Proof. ASSUME M is a map in the plane with five mutually adjacent countries. The five vertices in the dual $G(M)$, which correspond to the mutually adjacent countries in M, are mutually adjacent. So $G(M)$ contains K_5 as a subgraph. However, K_5 is not planar by Theorem 8.1.4, so $G(M)$ is not planar. But M is planar, so $G(M)$ is planar (see Note 8.2.B), a CONTRADICTION. So the assumption that a map has five mutually adjacent countries is false, and so not map has five mutually adjacent countries, as cliamed. □

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Theorem 8.3.2. The Five Color Theorem

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The countries of every normal map in the plane can be colored by five colors.

Proof. ASSUME that the theorem is false. That is, assume that there is a normal map that is not colorable by five colors. Of all such non five colorable maps, choose one with a minimal number of countries and denote it as M. We now consider six cases.

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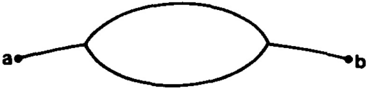

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Notice that vertices a and b must be distinct, or else M would contain a bridge and M would not be a normal map.

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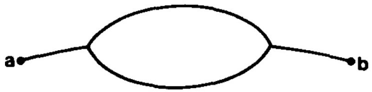

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Theorem 8.3.2. The Five Color Theorem (continued 1)

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The countries of every normal map in the plane can be colored by five colors.

Proof (continued). Consider the map M' that has the configuration of Figure 8.3.1 replace by an edge from a to b (so the lune is not present in M'), and the same as M everywhere else. Then M' is a normal map and the number of countries in M' is one less than the number of countries in M. Since M is has the minimum number of countries in a non five colorable normal map, then M' is colorable with five colors. In a coloration of M' by five colors, only two colors meet at the edge between a and b. Hence the lune can be reinserted and given one of the three other colors, giving a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a lune.

Theorem 8.3.2. The Five Color Theorem (continued 2)

Proof (continued). (b) Assume that M contains a triangular country, as in Figure 8.3.2.

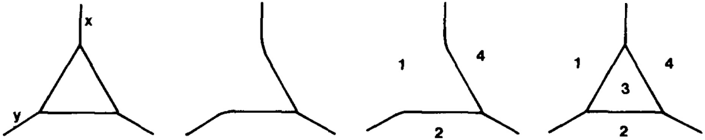

Figure 8.3.2

Notice that edges x and y must be distinct, or else M would contain a lune, in contradiction to case (a). Consider the map M' that is the same as M except that the triangular country is replaced by the second configuration in Figure 8.3.2. Then M' has one less than the number of countries in M. Since M has the minimum number of countries in a non five colorable normal map, then M' is colorable with five colors. For example, the countries of M' can be colored as given in the third part of Figure 8.3.2.

Theorem 8.3.2. The Five Color Theorem (continued 3)

Proof (continued). Since only three colors are involved here, we can reinsert the removed edge and then the original triangular country can be given one of the two other colors (as in the fourth part of Figure 8.3.2). This give a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a triangular country.

(c) Assume that M contains a country a that is a quadrilateral, as in Figure 8.3.3.

Theorem 8.3.2. The Five Color Theorem (continued 3)

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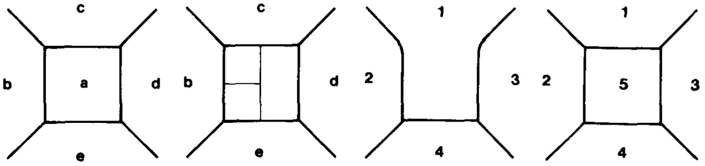

Figure 8.3.3

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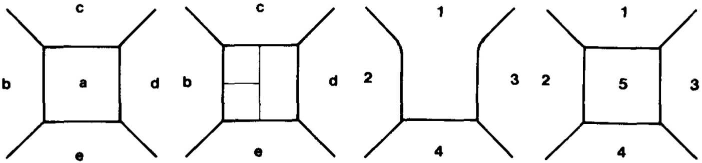

Figure 8.3.3

Theorem 8.3.2. The Five Color Theorem (continued 4)

Proof (continued). We claim that country a cannot have only two different countries adjacent to it and that either b and d are different, or c and e are different. For if this is not the case, we divide a into three countries, as in the second picture in Figure 8.3.3. Then we have five mutually adjacent countries in a planar normal map (the five mutually adjacent countries are the three new countries and countries $b = d$ and $c = e$) But this is not possible by Lemma 8.3.1, so we must have either b and d different or c and e different. So we can assume WLOG that c and e are different. Then we consider the map M' that results by deleting the edge between a and c, as given in the third picture of Figure 8.3.3. Then M' is a normal map with fewer countries than M. Since M has the minimum number of countries in a non five colorable normal map, then M' is colorable with five colors, as shown in the third picture of Figure 8.3.3. Since only four colors are involved here, we can reinsert the removed edge and then the original quadrilateral country can be given the other color (as in the fourth part of Figure 8.3.3).

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Proof (continued). We claim that country a cannot have only two different countries adjacent to it and that either b and d are different, or c and e are different. For if this is not the case, we divide a into three countries, as in the second picture in Figure 8.3.3. Then we have five mutually adjacent countries in a planar normal map (the five mutually adjacent countries are the three new countries and countries $b = d$ and $c = e$) But this is not possible by Lemma 8.3.1, so we must have either b and d different or c and e different. So we can assume WLOG that c and e are different. Then we consider the map M' that results by deleting the edge between a and c, as given in the third picture of Figure 8.3.3. Then M' is a normal map with fewer countries than M. Since M has the minimum number of countries in a non five colorable normal map, then M' is colorable with five colors, as shown in the third picture of Figure 8.3.3. Since only four colors are involved here, we can reinsert the removed edge and then the original quadrilateral country can be given the other color (as in the fourth part of Figure 8.3.3).

Theorem 8.3.2. The Five Color Theorem (continued 5)

Proof (continued). This gives a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a quadrilateral country.

(d) ASSUME that M contains a country that is a pentagon and is adjacent to five different countries.

Theorem 8.3.2. The Five Color Theorem (continued 5)

Proof (continued). This gives a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a quadrilateral country.

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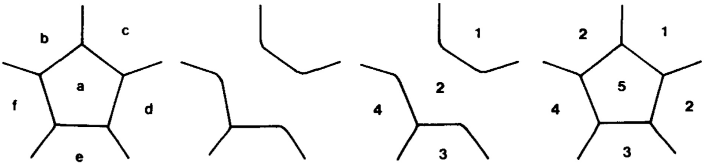

Figure 8.3.4

There must be a pair of these countries that is not adjacent, otherwise there would be five mutually adjacent countries contradicting Lemma 8.3.1. So we can assume WLOG that b and d are not adjacent.

Theorem 8.3.2. The Five Color Theorem (continued 5)

Proof (continued). This gives a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a quadrilateral country.

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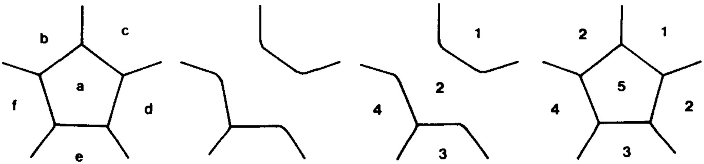

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Theorem 8.3.2. The Five Color Theorem (continued 6)

Proof (continued). Then we consider the map M' that results by deleting the edges between a and b and between a and d are deleted, as given in the second picture of Figure 8.3.4. Then M' is a normal map with fewer countries than M. Since M has the minimum number of countries in a non five colorable normal map, then M' is colorable with five colors, as shown in the third picture of Figure 8.3.4. Since only four colors are involved here, we can reinsert the removed edges and then the original quadrilateral country can be given the other color (as in the fourth part of Figure 8.3.4). This gives a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a country that is a pentagon and is adjacent to five different countries.

(e) ASSUME that M contains a country that is a pentagon and is adjacent to four different countries. See Figure 8.3.5 below.

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Proof (continued). Then we consider the map M' that results by deleting the edges between a and b and between a and d are deleted, as given in the second picture of Figure 8.3.4. Then M' is a normal map with fewer countries than M. Since M has the minimum number of countries in a non five colorable normal map, then M' is colorable with five colors, as shown in the third picture of Figure 8.3.4. Since only four colors are involved here, we can reinsert the removed edges and then the original quadrilateral country can be given the other color (as in the fourth part of Figure 8.3.4). This gives a five coloring of M, a CONTRADICTION to the fact that M was is a minimal normal map without a coloring using five colors. Therefore, M cannot contain a country that is a pentagon and is adjacent to five different countries.

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Theorem 8.3.2. The Five Color Theorem (continued 7)

Proof (continued). There must be a pair of these countries that is not adjacent, or else these four countries together with country a would be five mutually adjacent countries in contradiction to Lemma 8.3.1.

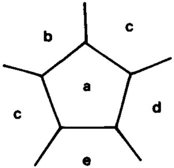

Figure 8.3.5

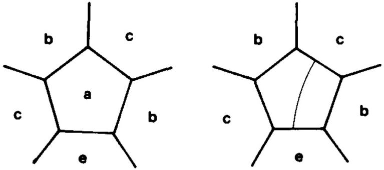

Figure 8.3.6

So we can assume WLOG that b and d are not adjacent. This case leads to a CONTRADICTION as in case (d) and Figure 8.3.4 above.

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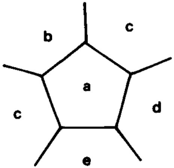

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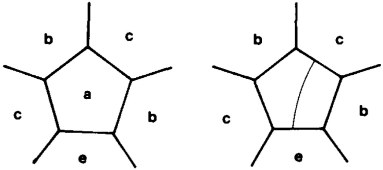

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So we can assume WLOG that b and d are not adjacent. This case leads to a CONTRADICTION as in case (d) and Figure 8.3.4 above.

(f) ASSUME that M contains a country that is a pentagon and is adjacent to three different countries. See Figure 8.3.6 above. We divide a into two countries, a_1 and a_2, as in Figure 8.3.6 right. This gives five mutually adjacent countries, in CONTRADICTION to Lemma 8.3.1.

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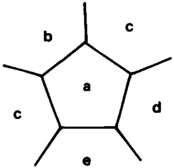

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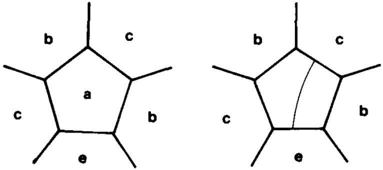

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Proof (continued). Thus, M does not contain this configuration.

We have covered all possible configurations involving a lune, triangle, quadrilateral, and pentagon in M (a minimal five colorable map), and each has lead to a contradiction. So map M contains no country smaller than a hexagon. Let p, q, and r be the number of vertices, edges, and countries in M, respectively.

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