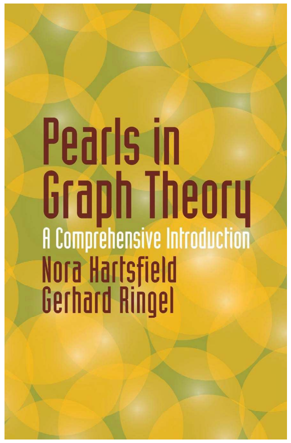
Introduction to Graph Theory

Chapter 9. Measurements of Closeness to Planarity

9.1. Crossing Number—Proofs of Theorems



Theorem 9.1.3

Theorem 9.1.3 Every simple drawing of K_4 in the plane has either zero or one crossing.

Proof. First, we observe that there are simple drawings of K_4 with one crossing and with no crossings, as shown in Figure 9.1.11.

Figure 9.1.11

We now prove that there is no more than one crossing by contradiction. In a simple drawing, suppose that one crossing is already present. ASSUME we can modify this drawing and introduce a second crossing in a modified simple drawing.

Lemma 9.1.3 (continued 1)

Theorem 9.1.3 Every simple drawing of K_4 in the plane has either zero or one crossing.

Proof (continued). We denote the edge with end vertices a and b as $(ab) = (b, a)$. WLOG, suppose the edges that result in the one crossing are (12) and (34), as shown in Figure 9.1.12.

Figure 9.1.11

Notice that in a simple drawing that edge (14) cannot cross edges (13), (12), (34), or (24) by part (b) of the definition of simple drawing. So if (14) crosses another edge then it must cross the edge (23). The boundary of the added triangle in Figure 9.1.11 divides the plane into an inside and an outside by the Jordan Curve Theorem (see Note 8.1.A).

Lemma 9.1.3 (continued 2)

Theorem 9.1.3 Every simple drawing of K_4 in the plane has either zero or one crossing.

Proof (continued).

Figure 9.1.11

So if the edge (14) crosses (23) then it cannot end at vertex 4, since (14) cannot exit the triangular region because, as argued above, it cannot cross (34) nor (12) and it cannot cross (23) a second time by part (a) of the definition of simple drawing, a CONTRADICTION. So the assumption that a simple drawing of K_4 can have a second crossing is false, and hence the number of crossings in a simple drawing of K_4 is either zero or one, as claimed. □

Theorem 9.1.4

Theorem 9.1.4 The crossing number of K_6 is $\text{cr}(K_6) = 3$.

Proof. First, we see from Figure 9.1.13 that $\text{cr}(K_6) \leq 3$. ASSUME there is a simple drawing of K_6 with only two crossings. Then there are two edges in K_6 whose removal results in a planar graph (notice that two crossings must involve at least three edges, by part (a) of the definition of simple drawing).

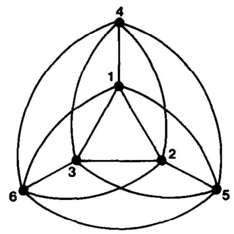


Figure 9.1.13

The resulting graph has $p = 6$ vertices and $q = 13$ edges. By Theorem 8.1.3 $q \leq 3p - 6$, but this implies $(13) \leq 3(6) - 6 = 12$, a CONTRADICTION. So the assumption that there is a simple drawing of K_6 with only two crossings is false. Therefore $\text{cr}(K_6) = 3$, as claimed. □

Theorem 9.1.5. Zarankiewicz's Theorem

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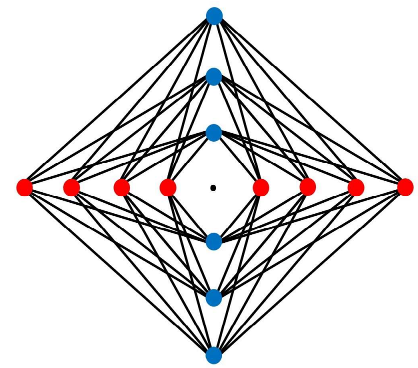
The crossing number of $K_{m,n}$ satisfies the inequality

$$\text{cr}(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor .$$

Proof. In order to get an upper bound for $\text{cr}(K_{m,n})$, consider the following figure (based on Figure 9.1.14) where there are m blue vertices on the vertical axis and n red vertices on the horizontal axis (for illustration, we take $m = 6$ and $n = 8$), evenly spaced and centered on the intersection of the axes. First suppose that m and n are both even, say $m = 2t$ and $n = 2s$.



Theorem 9.1.5. Zarankiewicz's Theorem (continued 1)

Proof (continued). Then there are t blue vertices on the top and t blue vertices on the bottom, and there are s red vertices on the left and s vertices on the right. Connect each red vertex to each blue vertex with a straight line, as given in the figure. Notice that if multiple crossings occur at the same point, then "we can simply move one of more vertices up or down a small amount" (Hartsfield and Ringel, page 185) so that the drawing is simple. In the first quadrant there is a crossing determined by each pair of red vertices to the right of the origin together with each pair of blue vertices above the origin (the edge joining the uppermost of the two blue vertices and the leftmost of the two red vertices crosses the edge joining the lowermost of the two blue vertices and the rightmost of the two red vertices). Thus there are $\binom{t}{2}\binom{s}{2}$ crossings in the first quadrant. Since all four quadrants have the same number of crossings by symmetry (since both m and n are even), then the total number of crossings is
$4\binom{t}{2}\binom{s}{2} = 4\frac{t(t-1)}{2}\frac{s(s-1)}{2} = t(t-1)s(s-1)$.

Theorem 9.1.5. Zarankiewicz's Theorem (continued 2)

Proof (continued). Thus this simple drawing shows that
$\text{cr}(K_{2t,2s}) \leq t(t-1)s(s-1) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

Next, suppose that m and n are both odd, say $m = 2t + 1$ and $n = 2s + 1$. We follow the same basic argument as when both m and n are odd. We place $t + 1$ blue vertices above the origin and t blue vertices below, and we place $s + 1$ red vertices to the right of the origin and s vertices to the left. We have lost the symmetry of the case when both m and n are even, and this time in quadrants I, II, III, and IV we have $\binom{t+1}{2}\binom{s+1}{2}$, $\binom{t+1}{2}\binom{s}{2}$, $\binom{t}{2}\binom{s}{2}$, and $\binom{t}{2}\binom{s+1}{2}$, respectively. So the total number of crossings is

$$\binom{t+1}{2}\binom{s+1}{2} + \binom{t+1}{2}\binom{s}{2} + \binom{t}{2}\binom{s}{2} + \binom{t}{2}\binom{s+1}{2}$$

$$= \left(\binom{t+1}{2} + \binom{t}{2}\right)\left(\binom{s+1}{2} + \binom{s}{2}\right)$$

Theorem 9.1.5. Zarankiewicz's Theorem (continued 3)

Proof (continued).

$$= \left(\frac{(t+1)t}{2}+\frac{t(t-1)}{2}\right)\left(\frac{(s+1)s}{2}+\frac{s(s-1)}{2}\right)$$
$$= \left(\frac{t(t+1+t-1)}{2}\right)\left(\frac{s(s+1+s-1)}{2}\right) = t^2s^2.$$

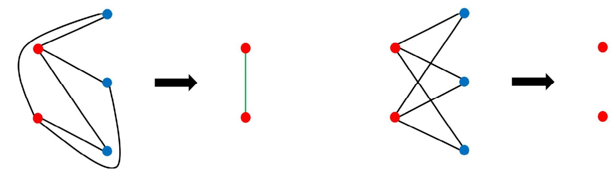
Again, the simple drawing shows that
$\mathrm{cr}(K_{2t+1,2s+1}) \le t^2s^2 = \left\lfloor \frac{m}{2}\right\rfloor \left\lfloor \frac{m-1}{2}\right\rfloor \left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$.

In the last case, we consider when one of m and n is even and the other is odd. WLOG, say $m = 2t+1$ and $n = 2s$. In Exercise 9.1.2 it is to be shown (using the same technique as used in the first two cases) that $\mathrm{cr}(K_{2t+1,2s}) \le t^2 s(s-1) = \left\lfloor \frac{m}{2}\right\rfloor \left\lfloor \frac{m-1}{2}\right\rfloor \left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$. Therefore, in all possible cases of the parities of m and n we have
$\mathrm{cr}(K_{m,n}) \le \left\lfloor \frac{m}{2}\right\rfloor \left\lfloor \frac{m-1}{2}\right\rfloor \left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$, as claimed. □

Theorem 9.1.6

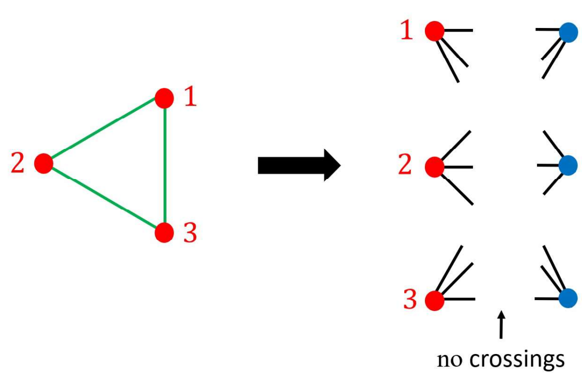
Theorem 9.1.6. The crossing number of $K_{3,n}$ is $\mathrm{cr}(K_{3,n}) = \left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$.

Proof. By Zarankiewicz's Theorem (Theorem 9.1.5), $\mathrm{cr}(K_{3,n}) \le \lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$. We show that in any simple drawing of $K_{3,n}$ that there are at least $\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$ crossings. ASSUME that we have a simple drawing of $K_{3,n}$ in the plane with a minimal number c of crossings. Say there are three blue vertices and n red vertices in the partite sets of $K_{3,n}$. The n red vertices will be the vertices of a new graph G. Consider any two red vertices of $K_{3,n}$; together with the three blue vertices they form a $K_{3,2}$. If there is *not* a crossing in this $K_{3,2}$ in the simple drawing of $K_{3,n}$, then we insert a green edge between the two red vertices in G.



Theorem 9.1.6 (continued 1)

Proof (continued). Thus G will have n (red) vertices and (as we go through all $\binom{n}{2}$ pairs of red vertices) at least $\binom{n}{2} - c$ edges ("at least" because a single $K_{3,2}$ could have more than one crossing). G has no triangles, since then there would be a drawing of $K_{3,3}$ in the plane with no crossings, contradicting Theorem 8.1.6.



Theorem 9.1.6 (continued 2)

Proof (continued). We have by Turan's Theorem (Theorem 4.1.2) with $k = 2$ (so that G contains no subgraph isomorphic to $K_{k+1} = K_3$), that the number of edges of G is at most equal to the number of edges of K_{n_1,n_2} where $n_1 + n_2 = n$ and $|n_1 - n_2| \le 1$. The number of edges of K_{n_1,n_2} is maximized for $n_1 = \lfloor (n+1)/2\rfloor$ and $n_2 = \lfloor n/2\rfloor$, when there are $\lfloor (n+1)/2\rfloor\lfloor n/2\rfloor$ edges. So the number of edges of G is at most $\lfloor (n+1)/2\rfloor\lfloor n/2\rfloor$. We now have that:

$$\binom{n}{2} - c \le \#\text{edges of } G \le \left\lfloor \frac{n+1}{2}\right\rfloor \left\lfloor \frac{n}{2}\right\rfloor.$$

This implies that $\binom{n}{2} - \left\lfloor \frac{n+1}{2}\right\rfloor \left\lfloor \frac{n}{2}\right\rfloor \le c$. By considering two cases based on the even/odd parity of n, it is straightforward to show that $\binom{n}{2} - \left\lfloor \frac{n+1}{2}\right\rfloor \left\lfloor \frac{n}{2}\right\rfloor = \left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$.

Theorem 9.1.6 (continued 3)

Theorem 9.1.6. The crossing number of $K_{3,n}$ is $\mathrm{cr}(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof (continued). Now $\binom{n}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \le c$ and $\binom{n}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ imply that $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \le c$. Therefore $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \le \mathrm{cr}(K_{3,n})$. Since we have already shown that $\mathrm{cr}(K_{3,n}) \le \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ by Zarankiewicz's Theorem (Theorem 9.1.5), then equality holds as claimed. □

Theorem 9.1.7

Theorem 9.1.7. The crossing number of $K_{4,n}$ is $\mathrm{cr}(K_{4,n}) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. By Zarankiewicz's Theorem (Theorem 9.1.5) we have $\mathrm{cr}(K_{4,n}) \le 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. We show that any simple drawing of $K_{4,n}$ has at least $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ crossings. Consider a simple drawing of $K_{4,n}$ with a minimal number of crossings. As above, let the four vertices of one partite set be blue and let the n vertices in the other partite set be red. Label the blue vertices 1, 2, 3, 4. Denote the number of crossing in the drawing that involve vertices 1 and 2 by h_{12}, those that involve 1 and 3 by h_{13}, and so forth. There are $\binom{4}{2} = 6$ such types of crossing and $h_{12} + h_{13} + h_{14} + h_{23} + h_{24} + h_{34} = \mathrm{cr}(K_{4,n})$. If we remove vertex 1 from the drawing, we are left with a simple drawing of $K_{3,n}$ with $h_{23} + h_{24} + h_{34}$ crossings.

Theorem 9.1.7 (continued)

Proof (continued). We get three more expressions for crossings in $K_{3,n}$ by removing one of the other vertices 2, 3, 4 from the original drawing. This produces four inequalities:

$$\begin{array}{ccccccccccccc}
 & & & & & & h_{23} & + & h_{24} & + & h_{34} & \ge & \mathrm{cr}(K_{3,n}) \\
 & & h_{13} & + & h_{14} & & & & & + & h_{34} & \ge & \mathrm{cr}(K_{3,n}) \\
h_{12} & & & + & h_{14} & & & + & h_{24} & & & \ge & \mathrm{cr}(K_{3,n}) \\
h_{12} & + & h_{13} & & & & h_{23} & & & & & \ge & \mathrm{cr}(K_{3,n}) \\
\hline
2(h_{12} & + & h_{13} & + & h_{14} & + & h_{23} & + & h_{24} & + & h_{34}) & \ge & 4\mathrm{cr}(K_{3,n})
\end{array}$$

Therefore we have $\mathrm{cr}(K_{4,n}) \ge 2\mathrm{cr}(K_{3,n}) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. Since we have already shown that $\mathrm{cr}(K_{4,n}) \le 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ by Zarankiewicz's Theorem (Theorem 9.1.5), then equality holds as claimed. □