Chapter 1. Basic Graph Theory

Section 1.1. Graphs and Degrees of Vertices

Note. In this section, we give several introductory definitions and introduce notation that we will use throughout. We state and prove two theorems.

Definition. A *graph* G is a pair of sets (V, E) where V is nonempty, and E is a (possibly empty) set of unordered pairs of distinct elements of V. The elements of V are called the *vertices* of G and the elements of E are called the *edges* of G. We represent the edge associated with the pair of vertices x and y as "xy." A graph is *finite* if its vertex set is finite and a graph is *infinite* if its vertex set is infinite. (We almost exclusive consider finite graphs.)

Note. We usually represent the vertices of a graph by points in the plane and represent edges (i.e., ordered pairs of vertices) as a curve (often a straight line segment) in the plane joining the two vertices of the edge. In this course, we use the term "graph" to mean finite graph, unless stated otherwise. A representation of a graph is given in Figure 1.1.1.

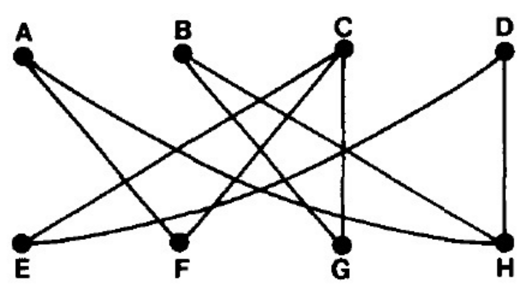


Figure 1.1.1

Note. Since a graph simple consists of objects (vertices) "connected" to other objects, we can use graphs to represent many physical situations. For example, a molecule can be represented by using vertices to represent atoms and edges to represent chemical bonds. See Figures 1.1.3 and 1.1.4.

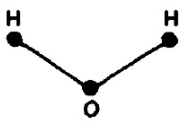


Figure 1.1.3. Water, H_2O.

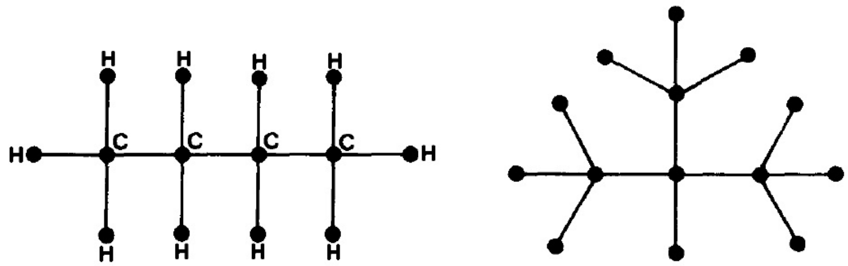
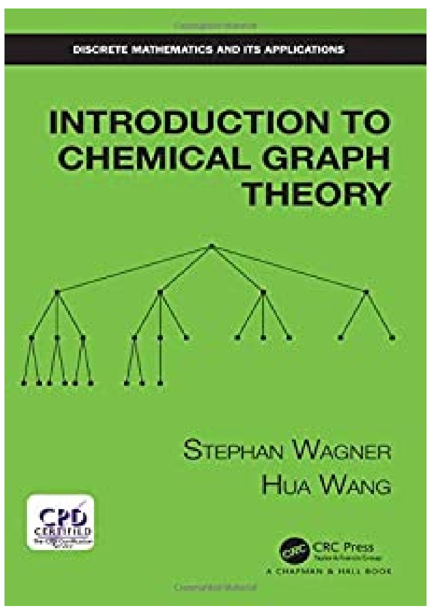
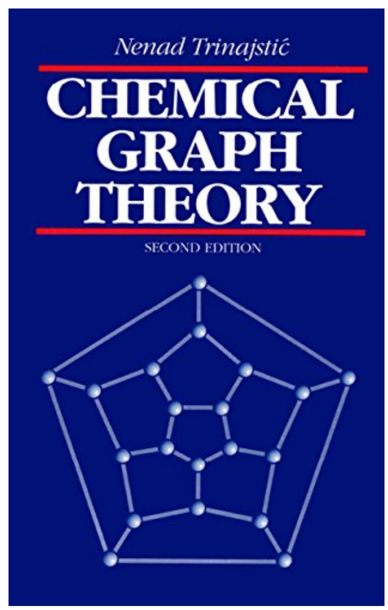


Figure 1.1.4. Butane and isobutane, C_4H_{10}.

In fact, chemical graph theory is an area of study in itself:





Note. Since two atoms can form multiple chemical bonds with each other (double bonds between carbon atoms are common, for example) it becomes necessary to address associated graphs which have more than one edge between two given vertices. See Figures 1.1.6 and 1.1.7 below. This inspires the next definition.

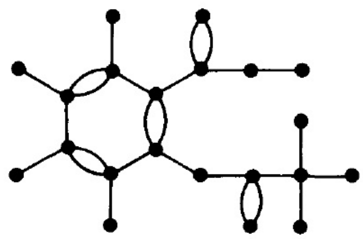


Figure 1.1.6. Aspirin, $C_9H_8O_4$.

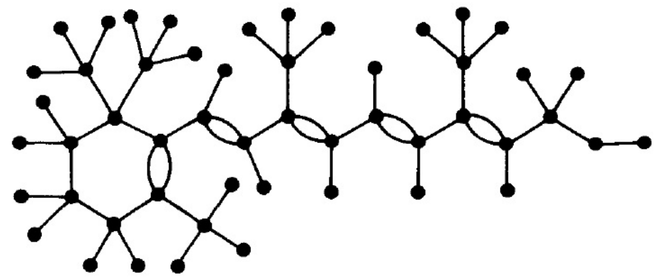


Figure 1.1.7. Vitamin A, $C_{20}H_{30}O$.

Definition. A *multigraph* G is a pair of multisets (in a multiset, elements can be repeated more than once) (V, E) where V is nonempty, and E is a (possibly empty) multiset of unordered pairs of elements of V. The elements of V are called the *vertices* of G and the elements of E are called the *edges* of G. A *loop* is a pair of vertices of the form (v, v). A graph or multigraph with loops added to the edge set/multiset is a *pseudograph*.

Note. Figures 1.1.6 and 1.1.7 show multigraphs, since they include several edges which are repeated twice (here, they represent double carbon bonds or, twice in the case of aspirin, double oxygen bonds). **In this book**, when we use the term "graph" we mean a structure as defined above which excludes repeated edges and excludes loops (so that a multigraph is not a graph, but a separate type of structure). **This is not a universal convention!** In fact, in the text used in our graduate-only Graph Theory 1 and 2 (MATH 5340 and MATH 5450), J.A. Bondy and U.S.R. Murty's *Graph Theory* (Graduate Texts in Mathematics #244, Springer, 2008), the term "graph" is used to represent what our text is calling a graph, a multigraph, and a pseudograph. The figure below is from this reference and gives a graph with vertex set $\{u, v, w, x, y\}$ and edge set $E = \{a = uv, b = uu, c = vw, d = xw, e = vx, f = xw, g = xy\}$ (here we the edges are then selves given names where represent the pairs of vertices).

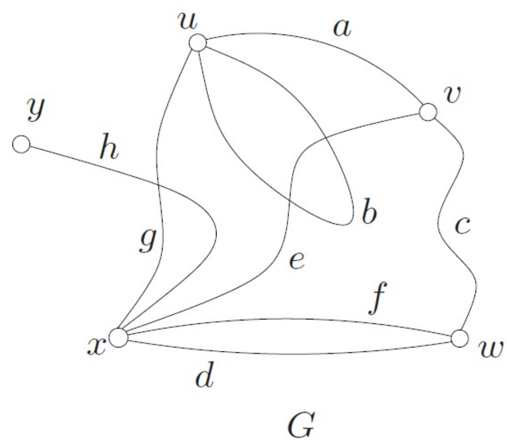


Figure 1.1(a) from Bondy and Murty's *Graph Theory.*

Definition. If x and y are vertices of a graph G, then x is *adjacent* to y if there is an edge between x and y. Adjacent vertices are called *neighbors.* The set of neighbors of vertex x is the *neighborhood* of x, denoted $N(x)$. Vertex x is *incident* with edge e if x is an endpoint of e (that is, x is one of the vertices in the pair of vertices that determine e). Edge e is *incident* with vertex x whenever x is an endpoint of e. The *degree* of a vertex v is the number of edges incident with v, denoted $d(v)$. A vertex of degree 0 is an *isolated vertex.* A vertex of degree 1 is an *end vertex.*

Note. In Figure 1.1.20, vertices a, d, and e are of degree 2, vertex b is of degree 3, and vertex c is of degree 1 (so c is an end vertex). The neighbors of vertex b are a, c, e so that the neighborhood of b is $N(b) = \{a, c, e\}$. Vertex v is incident with edges ab, bc, and be. Edge be is incident with vertices b and e. Vertices a and d are adjacent.

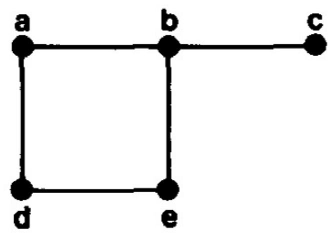


Figure 1.1.20

Note. We now state our first theorem. The proof is based on an easy counting argument.

Theorem 1.1.1. Let $v_1, v_2, \ldots, v_p$ be the vertices of a graph G, and let $d_1, d_2, \ldots, d_p$ be the degrees of the vertices, respectively. Let q be the number of edges of G. Then

$$d_1 + d_2 + \cdots + d_p = \sum_{i=1}^{p} d_i = 2q.$$

Note 1.1.A. Theorem 1.1.1 is sometimes called "The Handshaking Lemma" since it implies that in any group of people, an even number of people must have shaken the hands of an odd number of other people. More specifically, in any graph the number of vertices of odd degree must be even.

Definition. The *degree sequence* of a graph G is the list of non-negative integers that represent the degrees of the (distinct) vertices of G. A sequence of non-negative integers is called *graphic* if there exists a graph whose degree sequence is precisely that sequence.

Note. The degree sequence for the graph in Figure 1.1.20 above is 3, 2, 2, 2, 1. By convention, we will list the non-negative integers in a degree sequence in decreasing size. The next theorem gives a result that can by used to produce an algorithm to test a sequence to see if it is graphic.

Theorem 1.1.2. (Havel, Hakimi) Consider the following two sequences and assume sequence (1) is in descending order.

(1) $s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_n$

(2) $t_1 - 1, t_2 - 1, \ldots t_2 - 1, d_1, d_2, \ldots, d_n$.

The sequence (1) is graphic if and only if sequence (2) is graphic.

Exercise 1.1.4(d). Is the sequence 7 6 5 4 4 3 2 1 graphic?

Solution. We apply Theorem 1.1.2 to simplify the sequence. In the notation of Theorem 1.1.2, we must have $s = 7$. Now vertex S is adjacent to 7 vertices, so label these vertices $T_1, T_2, \ldots, T_7$ (we index these vertices in terms of decreasing degree so that $t_1 = 6$, $t_2 = 5$, $t_3 = 4$, $t_4 = 4$, $t_5 = 3$, $t_6 = 2$, and $t_7 = 1$). Then the given sequence is graphic if and only if the sequence 5 4 3 3 2 1 1 is graphic. But if this sequence is graphic, then we would have a graph with five vertices of odd degree, in violation of Theorem 1.1.1 (see Note 1.1.A). So NO the sequence is not graphic (this is where the fact that Theorem 1.1.2 is an “if and only if” result is useful). □

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