Chapter 1. Basic Graph Theory

Section 1.2. Subgraphs, Isomorphic Graphs

Note. In this section, we give a few more definitions and introduce some classes of graphs.

Definition. A subgraph of a graph $G$ is a graph $H$ such that every vertex of $H$ is a vertex of $G$, and every edge of $H$ is an edge of $G$ also or, symbolically, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Note. In Figure 1.2.1, three subgraphs of graph $G$ are given.

![Figure 1.2.1](image)

Note. The structure of a graph is given by the number of vertices and the way in which the vertices are connected by the edges. The vertices have not particular location nor do the edges have any particular shape. In Figure 1.2.2, two representations (i.e., two “drawings”) are given of the same graph. We are interested in
determining when two graphs are structurally the same. This is accomplished in the next definition.

\[
\begin{align*}
&\text{Definition.} \quad \text{Two graphs } G_1 \text{ and } G_2 \text{ with } p \text{ vertices are isomorphic if the vertices of } G_1 \text{ and } G_2 \text{ can be labeled with the numbers from 1 to } p \text{ such that whenever vertex } i \text{ is adjacent to vertex } j \text{ in } G_1, \text{ then vertex } i \text{ is adjacent to vertex } j \text{ in } G_2 \text{ and conversely. Such a labeling is the same as a one-to-one correspondence (a “bijection”) between } V(G_1) \text{ and } V(G_2) \text{ that preserves adjacency. More formally still, an isomorphism is a bijection } \alpha : V(G_1) \rightarrow V(G_2) \text{ such that } uv \text{ is an edge of } G_1 \text{ if and only if } \alpha(u)\alpha(v) \text{ is an edge of } G_2. \\
\end{align*}
\]

\[
\text{Figure 1.2.2}
\]

\[
\begin{align*}
&\text{Note.} \quad \text{Since an isomorphism preserves adjacency, then two isomorphic graphs must have the same number of vertices, the same number of edges, and the same degree sequences. In Figure 1.2.3, two graphs are given which have all three of these things in common. However, the graphs are still not isomorphic. Each graph has four vertices of degree 4. But in the graph of the left, each vertex of degree 4 is adjacent to one other vertex of degree 4, whereas in the graph on the right each vertex of degree 4 is adjacent to two other vertices of degree 4.}
\end{align*}
\]
1.2. Subgraphs, Isomorphic Graphs

Definition. The Petersen graph is the graph given in Figure 1.1.13 below. The complete graph on $n$ vertices is the graph on $n$ vertices in which every pair of vertices are adjacent, denoted $K_n$. The complete bipartite graph $K_{m,n}$ is the graph on $m + n$ vertices, where the vertex set can be partitioned into two sets $V_m$ and $V_n$ with $|V_m| = m$ and $|V_n| = n$ such that every vertex in $V_m$ is adjacent to every vertex in $V_n$ and there are no other adjacencies.

Note. The complete graph $K_4$ is given in Figure 1.1.14 and the complete bipartite graph $K_{4,4}$ is given in Figure 1.2.5. Graphs based on the platonic solids are
the tetrahedron (which is the same as $K_4$), the cube $Q_3$, the octahedron $O$, the dodecahedron $D$, and the icosahedron; see Figure 1.2.5.

![Figure 1.1.14](image1)

![Figure 1.2.5](image2)

**Definition.** The graph with $n + 1$ vertices labeled $x_0, x_1, \ldots, x_n$ and edges $x_0x_1, x_1x_2, \ldots, x_{n-1}x_n$ is a *path of length* $n$, denoted $P_n$. In such a path, $x_0$ and $x_n$ are *end vertices* and we say that $x_0$ and $x_n$ are *connected by the path*. The graph with $n$ vertices $x_0, x_1, \ldots, x_{n-1}$ and edges $x_0x_1, x_1x_2, \ldots, x_{n-1}x_{n-1}, x_{n-1}x_0$ is a *cycle of length* $n$, denoted $C_n$.

**Note.** The notation used here for a path of length $n$ is not universal. For example, J.A. Bondy and U.S.R. Murty’s *Graph Theory* (Graduate Texts in Mathematics #244, Springer, 2008) uses the notation $P_n$ to indicate a path on $n$ vertices (and so a path of length $n - 1$).