Chapter 1. Basic Graph Theory

Section 1.2. Subgraphs, Isomorphic Graphs

Note. In this section, we give a few more definitions and introduce some classes of graphs.

Definition. A *subgraph* of a graph G is a graph H such that every vertex of H is a vertex of G, and every edge of H is an edge of G also or, symbolically, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Note. In Figure 1.2.1, three subgraphs of graph G are give.

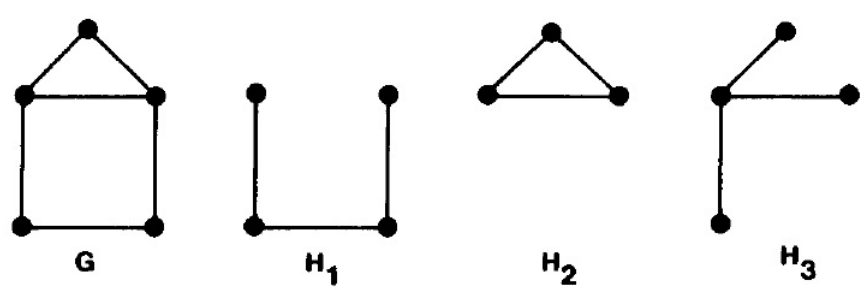

Figure 1.2.1

Note. The structure of a graph is given by the number of vertices and the way in which the vertices are connected by the edges. The vertices have not particular location nor do the edges have any particular shape. In Figure 1.2.2, two representations (i.e., two "drawings") are given of the same graph. We are interested in

determining when two graphs are structurally the same. This is accomplished in the next definition.

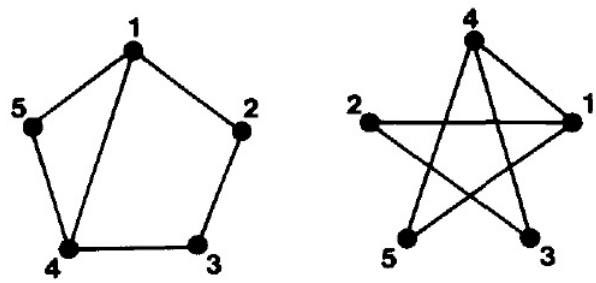

Figure 1.2.2

Definition. Two graphs G_1 and G_2 with p vertices are *isomorphic* if the vertices of G_1 and G_2 can be labeled with the numbers from 1 to p such that whenever vertex i is adjacent to vertex j in G_1, then vertex i is adjacent to vertex j in G_2 and conversely. Such a labeling is the same as a one-to-one correspondence (a "bijection") between $V(G_1)$ and $V(G_2)$ that preserves adjacency. More formally still, an isomorphism is a bijection $\alpha : V(G_1) \to V(G_2)$ such that uv is an edge of G_1 if and only if $\alpha(u)\alpha(v)$ is an edge of G_2.

Note. Since an isomorphism preserves adjacency, then two isomorphic graphs must have the same number of vertices, the same number of edges, and the same degree sequences. In Figure 1.2.3, two graphs are given which have all three of these things in common. However, the graphs are still not isomorphic. Each graph has four vertices of degree 4. But in the graph of the left, each vertex of degree 4 is adjacent to one other vertex of degree 4, whereas in the graph on the right each vertex of degree 4 is adjacent to two other vertices of degree 4.

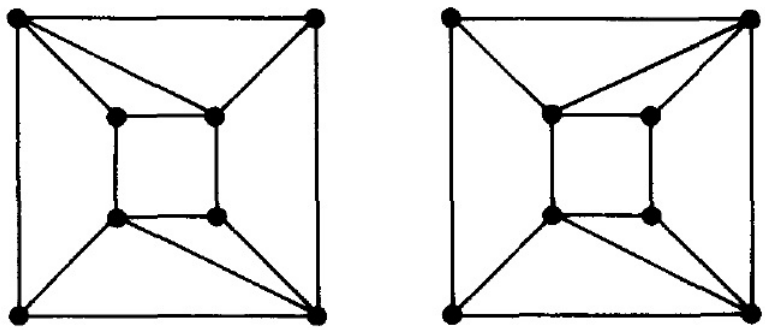

Figure 1.2.3

Definition. The *Petersen graph* is the graph given in Figure 1.1.13 below. The *complete graph* on n vertices is the graph on n vertices in which every pair of vertices are adjacent, denoted K_n. The *complete bipartite graph* $K_{m,n}$ is the graph on $m+n$ vertices, where the vertex set can be partitioned into two sets V_m and V_n with $|V_m| = m$ and $|V_n| = n$ such that every vertex in V_m is adjacent to every vertex in V_n and there are no other adjacencies.

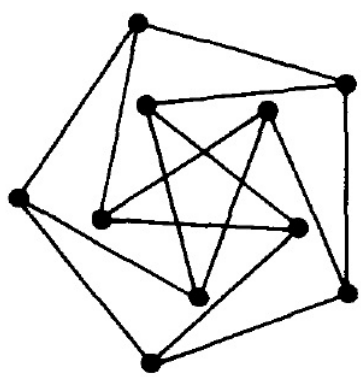

Figure 1.1.13

Note. The complete graph K_4 is given in Figure 1.1.14 and the complete bipartite graph $K_{4,4}$ is given in Figure 1.2.5. Graphs based on the platonic solids are

the tetrahedron (which is the same as K_4), the cube Q_3, the octahedron O, the dodecahedron D, and the icosahedron; see Figure 1.2.5.

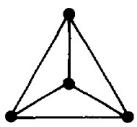

Figure 1.1.14

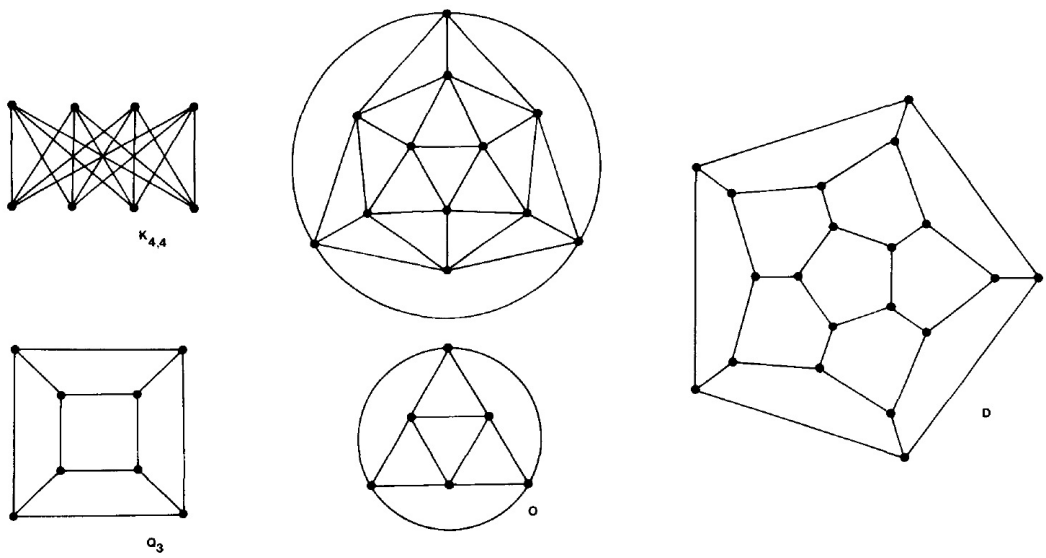

Figure 1.2.5

Definition. The graph with $n+1$ vertices labeled $x_0, x_1, \ldots, x_n$ and edges x_0x_1, $x_1x_2, \ldots, x_{n-1}x_n$ is a *path of length* n, denoted P_n. In such a path, x_0 and x_n are *end vertices* and we say that x_0 and x_n are *connected by the path.* The graph with n vertices $x_0, x_1, \ldots, x_{n-1}$ and edges $x_0x_1, x_1x_2, \ldots, x_{n-1}x_{n-1}, x_{n-1}x_0$ is a *cycle of length* n, denoted C_n.

Note. The notation used here for a path of length n is not universal. For example, J.A. Bondy and U.S.R. Murty's *Graph Theory* (Graduate Texts in Mathematics #244, Springer, 2008) uses the notation P_n to indicate a path on n vertices (and so a path of length $n-1$).

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