Chapter 1. Basic Graph Theory

Section 1.3. Trees

Note. In this section, we define trees, classify them in terms of the relationship between their number of vertices and edges, and address the connectedness of a graph in terms of certain subgraphs which are trees.

Definition. A graph G is *connected* if for any two vertices a and b there is a path between a and b in G.

Note. A necessary condition for graph G to be connected is given in the following theorem.

Theorem 1.3.1. If G is a connected graph with p vertices and q edges, then $p \leq q + 1$.

Note. We now define a new class of trees which play a role in classifying connected graphs (see Theorem 1.3.6).

Definition. A *tree* is a connected graph that contains no subgraph isomorphic to a cycle. A *forest* is a graph (connected or not) that contains no cycle.

Note. Figure 1.3.1 gives a forest containing several trees.

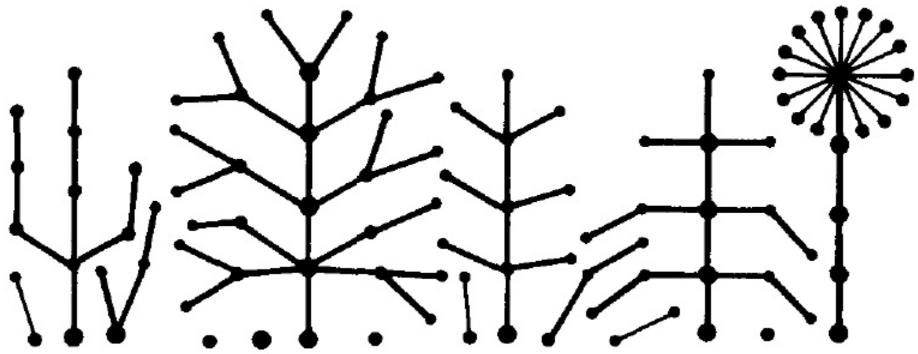

Figure 1.3.1 A forest

The next theorem gives a relationship between the number of vertices and the number of edges in a tree.

Theorem 1.3.2. If G is a tree with p vertices and q edges, then $p = q + 1$.

Note. The next theorem gives the conditions under which we have equality in Theorem 1.3.1.

Theorem 1.3.3. If G is connected, and $p = q + 1$, then G is a tree.

Note. We leave the proof of the next result as Exercise 1.3.3.

Theorem 1.3.4. Every tree with at least one edge has at least two end vertices.

Note. The text lists the following as an example, but we state it here as a theorem.

Theorem 1.3.A. Let the average degree of a connected graph G be greater than two. Then G has at least two cycles.

Note. An classification of trees in terms of subgraphs wich are paths is given in the next result.

Theorem 1.3.5. A graph G is a tree if and only if there exists exactly one path between any two vertices.

Definition. A *spanning subgraph* of a graph G is a subgraph H or G such that H contains all the vertices of G, so that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A *spanning tree* of G is a spanning subgraph of G that is a tree.

Note. The next result allows us to address the connectivity of a graph in terms of the existence of a spanning tree.

Theorem 1.3.6. Every connected graph G contains a spanning tree.

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