Chapter 10. Graphs on Surfaces

Note. In the first section of this chapter we define rotations of a graph and use it to count circuits in the graph symbolically, without an appeal to drawings of the graph. In the second section we use rotations and circuits to classify planar graphs without an appeal th drawings, crossing numbers, or the Jordan Curve Theorem. In the third section, we consider graphs on surfaces other than the plane, and use rotations and circuits to find the genus of a graph (that is, we find the genus of a surface in which the graph can be embedding without crossings). The book concludes with some final observations of the chromatic number of a graph.

Section 10.1. Rotations of Graphs

Note. In this section, we introduce a rotation of a graph and use it to find "circuits" in the graph. We give a symbolic way to represent rotations of a graph and use it to find circuits without appealing to drawings of the graph. For a given rotation of a graph, we give a relationship between the number of vertices, edges, and circuits of a graph (in Theorem 10.1.2). We start with cubic graphs now, but consider more general graph later in this section.

Definition. A *rotation*, denoted ρ, of a cubic graph G is an assignment of the color black or the color white to each vertex.

Note. The reason for the term "rotation" will become apparent. For a given drawing, we associate the clockwise direction with all black vertices and the counterclockwise direction with all white vertices, as given in the two diagrams on the left of Figure 10.1.1.

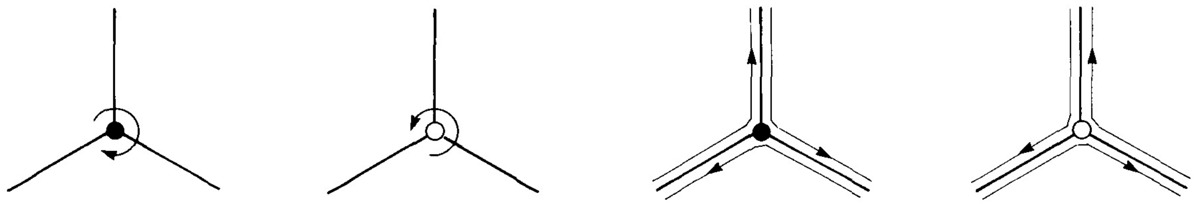


Figure 10.1.1

The two diagrams on the right of Figure 10.1.1 are to be interpreted as follows. If you travel along an edge of the cubic graph (in either of the two possible directions) then after you reach a vertex, there are two options to continue onward. One is to your left and the other is two your right. So when you travel to a black vertex turn to your left at the vertex, and when you travel to a white vertex turn to your right at the vertex.

Definition. If, in following a rotation through a cubic graph, you return to your starting vertex in such a way that the next edge would be the first edge repeated in the same direction, then the journey has determined a *circuit*. The circuit is *induced* by the rotation. (Notice that a circuit consists of edges here, though you might use the term "circuit" in reference to arcs elsewhere).

Note 10.1.A. A rotation ρ of a cubic graph is given in Figure 10.1.2 (left). There are three circuits induced by ρ (Figure 10.1.2 right and Figure 10.1.3, below). We denote the number of induced circuits of a rotation ρ as $r(\rho)$, so here we have $r(\rho) = 3$. Notice that the circuit in Figure 10.1.3 left is not a cycle, since it repeats one edge. Notice that each edge appears exactly twice in the collection of circuits (once in each direction).

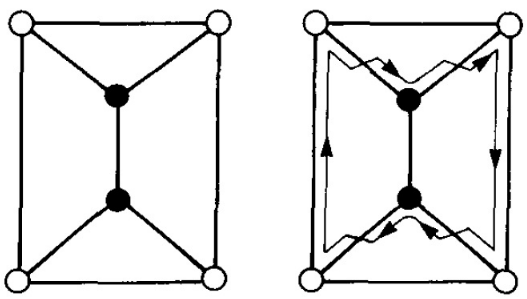


Figure 10.1.2

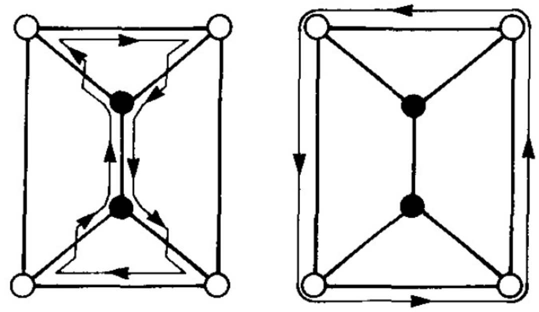


Figure 10.1.3

Note. Another rotation, ρ_1, of the same graph given in Figure 10.1.2. In this case, there is only one circuit induced by the rotation, so that $r(\rho_1) = 1$.

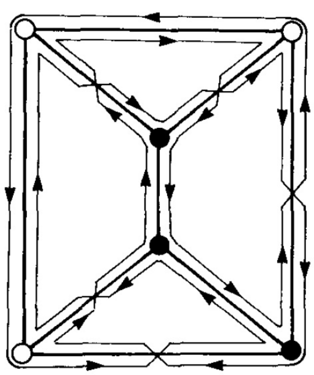
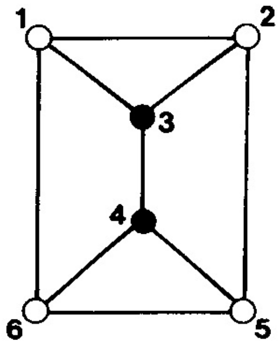


Figure 10.1.4

Definition. A rotation of a graph which induces only one circuit is a *circular rotation.*

Note. We can describe a graph by numbering the vertices and then, for each vertex, listing the vertices adjacent to that vertex. This is called a "scheme" for the graph. Figure 10.1.8 gives a cubic graph and one of its schemes.

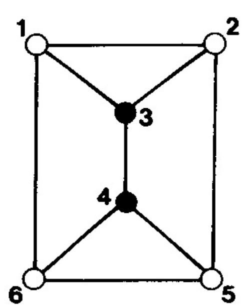


1. 2, 3, 6
2. 1, 3, 5
3. 1, 2, 4
4. 3, 5, 6
5. 2, 4, 6
6. 1, 4, 5

Figure 10.1.8

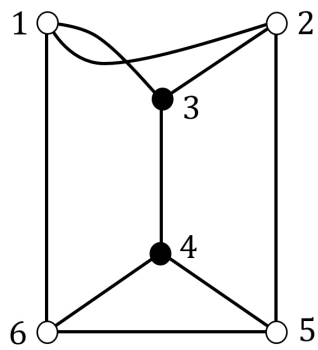
Notice that if the ith row contains j, then the jth row must contain i (because i and j are adjacent to each other). In fact, any list with this property describes a graph.

Note 10.1.B. Notice that the order in which the vertices appear in a row of a scheme is irrelevant. So we can list the vertices for a cubic graph in such a way as to indicate a rotation. In Figure 10.1.8, vertex 1 is white so it is treated as counterclockwise. Reading off the vertices incident to vertex 1 in counterclockwise order gives 6 3 2 (or 3 2 6 or 2 6 3; these three triples are "cyclic permutations" of each other). In this way, a graph and a rotation can be given without a drawing necessary. A scheme for the graph of Figure 10.1.8 which reflects the rotation is:



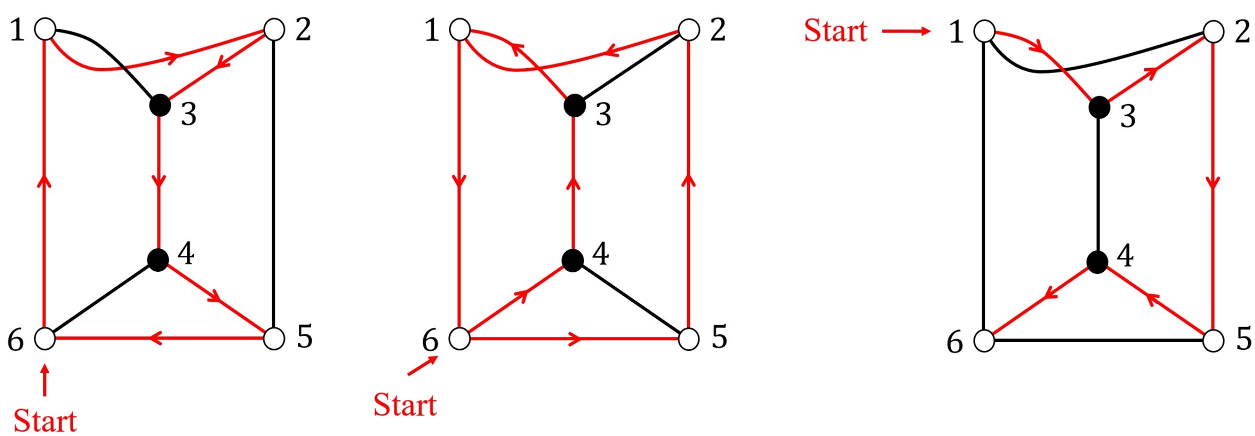
1. 6 3 2
2. 1 3 5
3. 1 2 4
4. 3 5 6
5. 2 4 6
6. 5 4 1

However, the drawing affects this type of scheme. Here is another drawing of the same graph with the same rotation, but the scheme is different, since it is different at vertex 1:



1: 6 2 3
2: 1 3 5
3: 1 2 4
4: 3 5 6
5: 2 4 6
6: 5 4 1

The drawing also affects the circuits and the number of circuits. We saw in Figures 10.1.2 and 10.1.3 that, when drawn without any crossings, the graph has three circuits with respect to the indicated rotation, $r(\rho) = 3$. If we take the same graph with the same rotation, but consider the drawing with a crossing, we find that there is only one circuit. Starting at vertex 6 and breaking the circuit into three pieces for clarity, we have:



You will find if you use this same drawing, but modify the rotation so that vertex 1 is black, that there are three circuits that result (the same three circuits given in Figures 10.1.2 and 10.1.3).

Definition. A *rotation* of a vertex in a (not necessarily cubic) graph is (any cyclic permutation of) an ordering of the vertices adjacent to that vertex. A *rotation* of a graph consists of rotations of every vertex of the graph.

Note. For a cubic graph, there are only two rotations for a given vertex, a "clockwise" rotation and a "counterclockwise" rotation. If we drop these labels of "clockwise" and "counterclockwise" (which are dependent on the drawing), we can simply use the ordering to determine the rotation of a vertex in a cubic graph. If the neighbors of a vertex are a, b, c then the two rotations are $a\,b\,c$ and $a\,c\,b$ (either of which can also be represented by a cyclic permutation of the given three vertices). Just as with cubic graphs, we can use a scheme to represent a non-cubic graph and can represent the rotation of each vertex by ordering the vertices. Since we can use any ordering of the neighbors of a vertex, there are many possible rotations for a vertex of high degree. Quantitatively, we have the next result.

Theorem 10.1.1. If a vertex v of a graph has degree d, then there are $(d-1)!$ different rotations of v.

Note 10.1.C. By definition of a rotation, no edge is mapped to itself by a rotation at a vertex unless the vertex is degree one.

Note. We can use the rotation of a graph to find circuits, without an appeal to a drawing. We present the rotation using a scheme that gives the rotation in terms of the order in which neighbors are given. For K_5 with vertex set $\{0, 1, 2, 3, 4\}$, consider the scheme and rotation:

0.	1	3	2	4
1.	3	0	2	4
2.	1	4	0	3
3.	2	4	1	0
4.	1	0	3	2

Starting at vertex 0 (say) we choose any neighbor, say 1. That is, we follow edge 0 1 from vertex 0 to vertex 1. In row 1 we see that after 0 (the vertex we are coming from) there is a 2, so next we follow edge 1 2 from vertex 1 to vertex 2. Similarly, we consider row 2 and see that after 1 there is a 4, so next we follow edge 2 4 In row 4, 2 is followed by 1 so we follow edges 4 1. We continue in the way until we return to vertex 0 *and* do so in such a we that we would next follow edge 0 1 (from row 1 this means that we must have come from vertex 4 to vertex 0 in the preceding step before we stop). The vertices in the circuit are, in order: 0 1 2 4 1 3 0 2 3 4 2 0 4 3 1 0 3 2 1 4. Next we would go to vertex 0 and complete the circuit. Notice that there are 20 vertices here, and so 20 edges have been traverse. That is, each of the 10 edges of K_5 have been traversed in both directions, so there are no other circuits in K_5. With ρ as the rotation, we therefore have $r(\rho) = 1$ and ρ is a circular rotation.

Note. In the next section, we will define a planar graph in purely graph theoretic terms. We will use rotations and the number of circuits, which we now see can be handled in a mechanical way. The inequality in the next result will play a large role in this approach.

Theorem 10.1.2. Given a connected graph with p vertices and q edges, and a rotation ρ which induces $r(\rho)$ circuits, the inequality $p - q + r(\rho) \leq 2$ holds. Furthermore, the alternating sum $p - q + r(\rho)$ is even.

Note. Special things happen when equality holds in the inequality of Theorem 10.1.2. On to Secion 10.2. . .

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