Section 10.2. Planar Graphs Revisited

Note. In this section, we give an alternate definition of a planar graph using rotations from the previous section. We use this new definition to establish some results that were already known based on the definition of planar graph from Section 8.1. Planar Graphs. We show that previous definition of planar implies the new definition (in Theorem 10.2.9), but we do not show the converse since it requires more topology than we likely know.

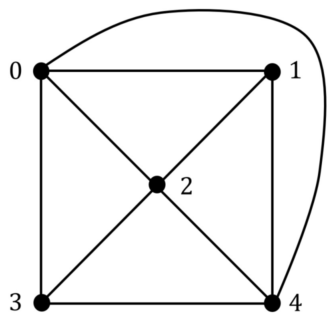
Definition. A connected graph G with p vertices and q edges for which there exists a rotation ρ of G such that $p - q + r(\rho) = 2$ is a *connected planar graph*. Such a rotation ρ is a *planar rotation* of G.

Note 10.2.A. Not all rotations of a planar graph are planar rotations. The graph $K_5 - e$ is planar under this new definition (consider a rotation that is clockwise at each vertex, for example, in the next figure left). However, the scheme and rotation:

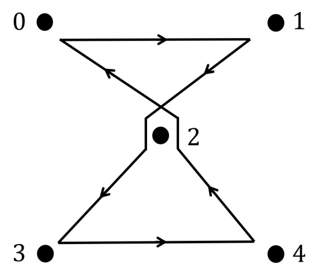
0.	2	1	4	3
1.	0	2	4	
2.	1	3	4	0
3.	2	4	0	
4.	3	2	1	0

This rotation produces the two circuits given below. Notice that one is of length 6 and the other is of length 12. We have $p = 5$, $q = 9$, and, for the given rotation ρ,

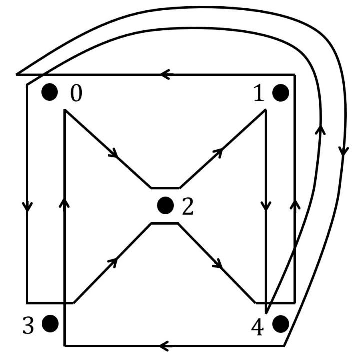
$r(\rho) = 2$. Hence $p - q + r(\rho) = 5 - 9 + 2 = -2$. So this rotation is not a planar rotation, even though $K_5 - e$ is planar.



$K_5 - e$



0 1 2 3 4 2



0 2 1 4 0 3 2 4 1 0 4 3

Definition. If G is not connected, then we can consider the connected components of G and apply the previous definition. If each component of G is planar, then we say that G itself is *planar*.

Note. In the proofs of the following, we rely somewhat on pictures. Arguably, this part of the presentation is less rigorous than much of the rest of the course. However, the pictures we present are used to illustrate the order of vertices in circuits. We don't use the pictures to reflect anything related to crossings of edges, since this section addresses planarity in terms of rotations alone.

Theorem 10.2.1. If a graph G has a bridge, then for any rotation ρ of G, the bridge occurs in both directions in one circuit induced by ρ.

Theorem 10.2.2. If H is a subgraph of a planar graph G, then H is planar.

Theorem 10.2.3. The complete bipartite graph $K_{3,3}$ is not planar.

Note. We can similarly show that K_5 is not planar, and the proof of this is left as Exercise 10.2.4.

Theorem 10.2.4. The complete graph K_5 is not planar.

Definition. A *maximal planar graph* is a planar graph with the property that if any pair of nonadjacent vertices are connected by an edge, the resulting graph is no longer planar.

Note 10.2.B. The graphs K_3 and K_4 are maximal planar graphs (since they are planar and have no pair of nonadjacent vertices). The graph $K_5 - e$ is a maximal planar graph, since there is only one pair of nonadjacent vertices, but an edge between these vertices cannot be added since this would result in the nonplanar K_5. The graph $K_{3,3} - e$ is planar, but not maximal since some edges can be added between vertices in the partite sets and the new graph will still be planar.

Theorem 10.2.5. A maximal planar graph G with three or more vertices is connected and has no bridge.

Note 10.2.C. Notice that Theorem 10.2.5 implies that a maximal planar graph (with three of more vertices) cannot contain a pendant edge, since a pendant edge is an example of a bridge. By Note 10.1.C, a pendant edge in a circuit occurs only when the edge is pendant in graph G.

Note. For the next theorem, we give the text book's proof. Your humble instructor understand the last step of the proof. This will be discussed below, after presenting the "proof."

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

Note. In the "proof" of Theorem 10.2.6, it is shown for maximal planar graph G with planar rotation ρ that: there is no circuit induced by ρ of length four that is a cycle (in Case 2), there is no circuit of length five or more induced by rotation ρ with all vertices distinct (in Case 1), and there is no circuit of length at least eight with a repeated edge (in Case 3). From this, the text concludes that every circuit induced by ρ has length three. It is unclear how this conclusion can be drawn. We know that a maximal planar graph can have *some* rotation with circuits longer than length three, as given in Note 10.2.A (in fact, there is a 12 cycle which repeats edges); of course, the rotation given is not a *planar rotation*. So somehow the fact that ρ is a planar rotation must be used in drawing this conclusion. Unfortunately, your "humble instructor" does not see how to justify the final line of the proof.

Note. By Note 10.2.B, we have that $K_5 - e$ is a maximal planar graph. By Note 10.2.A we see that there is a rotation (but not a *planar rotation*) of $K_5 - e$ with circuits longer than length three.

Theorem 10.2.7. In a maximal planar graph with p vertices, $p \geq 3$, and q edges we have $q = 3p - 6$.

Theorem 10.2.8. Suppose that G is a maximal planar graph with p vertices, $p \geq 4$. Let p_i denote the number of vertices of G with degree i. Then

$$3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + 3p_9 + \cdots.$$

Note. Theorem 10.2.8 is the same as Theorem 8.1.8, but is based on the new definition of planar from this section. The proof of Theorem 10.2.8 is the same as the proof of Theorem 8.1.8, with the exception that the use of Theorem 8.1.2 in the proof of Theorem 8.1.8 is replaced with Theorem 10.2.7.

Note. We would like to prove that the previous definition of a planar graph (in terms of drawings with no crossings) is equivalent to the definition in this section (in terms of $p - q + r(\rho) = 2$ for the connected components of the graph). In the next theorem, we prove that the previous definition implies the definition in this section. To prove the converse, that the definition of this section implies the previous definition (and, hence, conclude that the two definitions are equivalent), requires an exploration of topology that is "beyond the scope of this course."

Theorem 10.2.9. A connected graph that can be drawn in the plane with no crossings has a planar rotation. That is, graphs that are planar under the definition in Chapter 8 are also planar under the definition in Chapter 10.

Note. The graph of Figure 10.2.5 is planar, but it is not maximal. We can add additional edges to the graph until all circuits are of length three (and all faces are bounded by three edges, even the outer face), as follows:

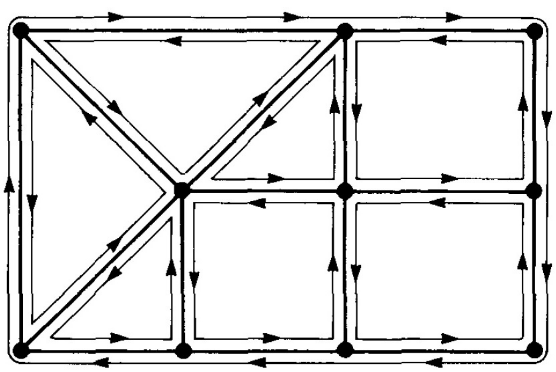
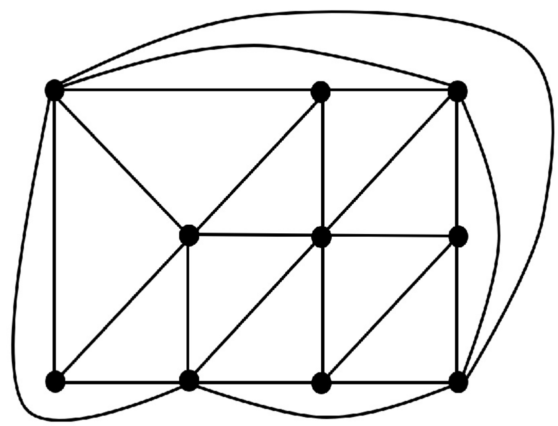


Figure 10.2.5



So the graph on the right here is a maximal planar graph. This drawing is called a "triangulation" of the maximal planar graph because each face has three edges. Triangulations are considered in graduate Graph Theory 2 (MATH 5450) in Section 10.2. Duality.

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