Section 10.3. The Genus of a Graph

Note. In this section, we address graphs on surfaces using the idea of rotations, first introduced in Section 10.1. We briefly consider orientable closed surfaces of genus g, though our examples mostly concentrate on the torus which is genus 1. We define the genus of a graph in two ways and study the genus of complete graphs and complete bipartite graphs. These topics are also covered in Graph Theory 2 (MATH 5450) in Section 10.6. Surface Embeddings of Graphs where nonorientable surfaces are also discussed.

Note 10.3.A. If a graph can be drawn in the plane with no crossings, then it can be drawn on the sphere with no crossings (and conversely). The reason for this is that we can project the drawing from the plane to the sphere (or back) using the *stereographic projection* of all but one point on the sphere onto the plane, as in the following figure.

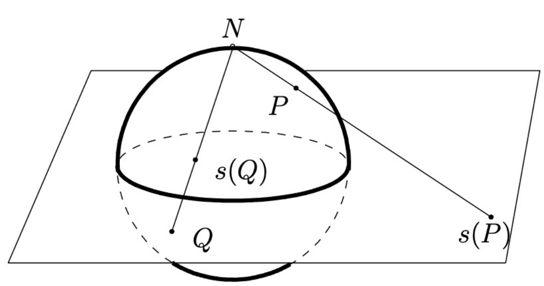

Image from the LibreTexts Mathematics,
1.3: Stereographic Projection webpage (accessed 2/1/2023).

The sphere is an example of a "closed surface" or a "surface with no boundary." Another example is a torus. Examples of surfaces that are *not* of this type include a disk or a hemisphere (since both have a boundary). We shall refer to closed surfaces simply as "surfaces," and not explore surfaces very deeply but mostly rely on informal descriptions and pictures of surfaces. Formally, a surface is a "2-manifold"; an n-manifold is sort of an n-dimensional surface. A quick (but rigorous) introduction to surfaces can be found in my online notes for Differential Geometry (with an emphasis on relativity; MATH 5310) on Section 1.9. Manifolds. A thorough coverage of n-manifolds is given in some other online notes I have for Differential Geometry on Section VII.2. Manifolds. Complex manifolds (manifolds based on complex number "patches") are addressed in my online notes for Complex Analysis 2 (MATH 5520) on Section IX. Analytic Manifolds. Of relevance to graph theory, is the classification of surfaces. This would be covered in a senior/graduate level Introduction to Topology 2 class. ETSU does not formally have such a class, but I have some online notes (with more in preparation) for this on Algebraic Topology (notice Chapter 12). An elementary, visual discussion of surfaces is given in my online presentation (in PowerPoint) The Big Bang and the Shape of Space. This presentation discusses what a "Flatland" character would experience as they traveled on various surfaces.

Note/Definition. Figure 10.3.1 is of a torus. It has different "connectivity" from that of a sphere. A surface with an even more complicated connectivity is given in Figure 10.3.2; this surface has three "handles" and is denoted S_3 in this section. Similarly, the sphere is denoted S_0 and a torus is denoted S_1 (since it is a sphere with one handle; or at least a sphere with one handle can be continuously deformed into a torus). In general, a surface that results from adding g handles to a sphere is denoted S_g is the *orientable surface of genus g*. We are interested in when a graph G can be drawn on surface S_g with no edge crossings (such a drawing is called an *embedding* of G in S_g).

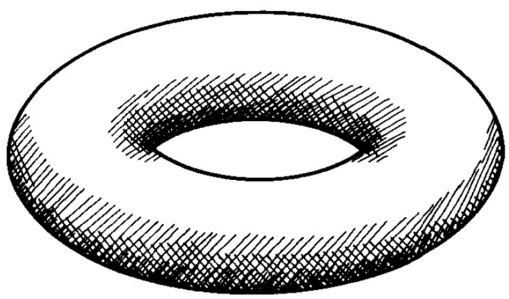

Figure 10.3.1

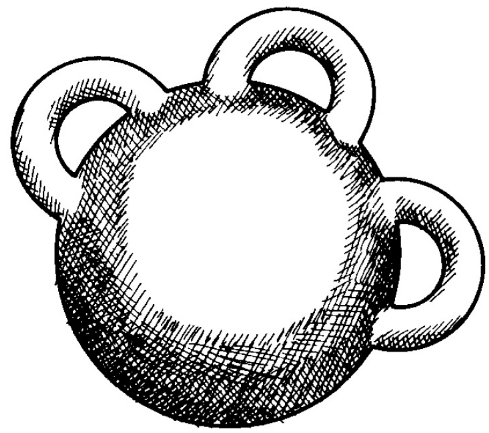

Figure 10.3.2

Note/Definition. If a graph is embeddable into a surface S_g, then it is also embeddable into S_{g+1} (just avoid the additional handle when drawing the graph on the more complicated surface). The minimum value of g for which graph G is embeddable in S_g is the *genus* of G, denoted $\gamma(G)$. We see from the stereographic projection, that any planar graph has genus 0. We know that K_5 and $K_{3,3}$ are nonplanar (by Theorems 8.1.4 and 8.1.6, respectively), so their genus is greater than 0. We'll see below that $\gamma(K_5) = \gamma(K_{3,3}) = 1$.

Note. A common way to represent a drawing of a graph on a torus is "peel apart" the torus into a flat square (called the "fundamental domain" of the surface). The drawing is then given on the square with the understanding that the sides are joined together, and the top and bottom are joined together, as indicated by the arrows in Figure 10.3.3 left.

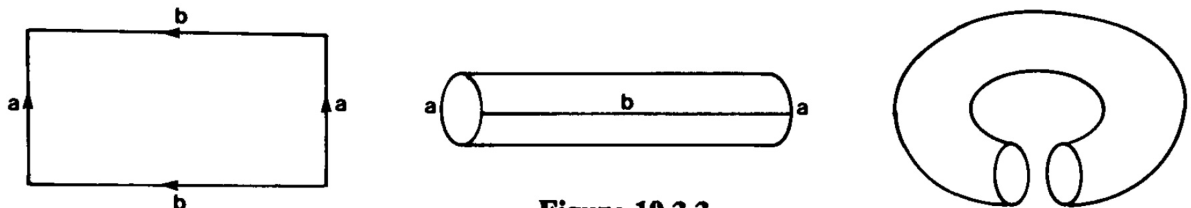

Figure 10.3.3

Note 10.3.B. Figure 10.3.4 gives two embeddings of K_5 on the torus. Notice that the edges at the top and bottom of Figure 10.3.4 left are actually two parts of the same edge (as is the case for the edges at the left and right sides). In Figure 10.3.4 right, the edge that exists the left side continues on as it enters the right side; there are only five vertices, but because of the connections of the fundamental domain the vertices 0, 1, 2, 3 are repeated twice and the vertex 4 is repeated four times. Figure 10.3.5 gives an embedding of $K_{3,3}$ on the torus. Notice that the little arc in the upper right hand corner is part of the edge joining 0 and 3.

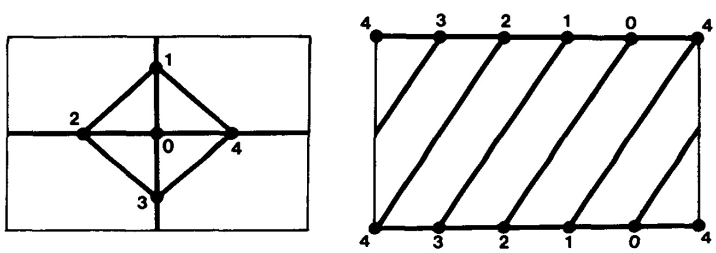

Figure 10.3.4

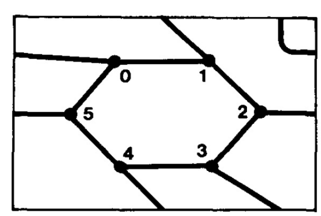

Figure 10.3.5

Note. A thorough exploration of graphs on surfaces and the genus of a graph would require a deep dive into topological graph theory and the topology of closed surfaces. A reference on this is Bojan Mohar and Carsten Thomassen's *Graphs on Surfaces*, (Johns Hopkins University Press, 2001). I have notes in preparation based on this source for Topological Graph Theory.

Note. For a connected graph G with p vertices, q edges, and rotation ρ, by Theorem 10.1.2 the number of circuits $r(\rho)$ induced by the rotation satisfies $p - q + r(\rho) \leq 2$. Also, $p - q + r(\rho)$ must be even. The proof of Theorem 10.1.2 is based only on connectivity (that is, the innate structure of the graph), and not on any surface on which the graph might be embedded.

Definition. For a given connected graph G, a rotation ρ of G is a *maximal rotation* of G if $r(\rho)$ is as large as possible over all rotations of G.

Note 10.3.C. If G has a planar rotation ρ (so that $p - q + r(\rho) = 2$), then ρ is a maximal rotation. If we consider K_5 then $p = 5$ and $q = 10$. So for any rotation of K_5 we have $p - q + r(\rho) = r(\rho) - 5 \leq 2$ (or $r(\rho) \leq 7$) and $r(\rho) - 5$ is even. Now $r(\rho) \neq 7$ because the shortest length a circuit can have is three (we could only have a circuit of length two for the graph K_2), and 7 circuits would require at least 21 edges (in both directions), but K_5 only has 10 edges (which give 20 edges "in both directions" for use in circuits). So we cannot have $r(\rho) - 5 = 2$ and it must be (because of the evenness of $r(\rho) - 5$) that $r(\rho) \leq 5$ for K_5. A rotation of K_5 with $r(\rho) = 5$ (hence a maximal rotation of K_5) is:

0.	1	4	3	2
1.	0	2	3	4
2.	0	3	4	1
3.	0	4	1	2
4.	0	1	2	3

The 5 circuits of this rotation are 0 1 2, 0 2 3, 0 3 4, 0 4 1, and 1 3 2 4 3 1 4 2. In fact, these circuits are precisely the boundaries of the faces of the embedding of K_5 in the torus given in Figure 10.3.4 left (above). Another maximal rotation of K_5 is:

0.	1 3 4 2
1.	2 4 0 3
2.	3 0 1 4
3.	4 1 2 0
4.	0 2 3 1

The 5 circuits of this rotation are 0 3 4 1, 0 2 1 4, 0 1 3 2, 0 4 2 3, 1 2 4 3. This rotation is given in Section 10.1 of the text book (see page 214). In fact, these circuits are precisely the boundaries of the faces of the embedding of K_5 in the torus given in Figure 10.3.4 right (above).

Note 10.3.D. If we consider $K_{3,3}$ then $p = 6$ and $q = 9$. So for any rotation ρ of $K_{3,3}$ we have by Theorem 10.1.2 that $p - q + r(\rho) = r(\rho) - 3 \leq 2$ (or $r(\rho) \leq 5$) and $r(\rho) - 3$ is even (or $r(\rho)$ is odd). Now $K_{3,3}$ has 9 edges (which give 18 edges "in both directions" for use in circuits). Since $K_{3,3}$ is bipartite, the length of each circuit must be even (and greater than two, as commented in Note 10.3.C). The largest number of elements of $\{4, 6, 8, 10, 12, 14, 16, 18\}$ which sum to 18 is four (consider $4+4+4+6 = 18$). However $r(\rho) = 4$ is even, hence $r(\rho) \leq 3$. With the labelings of the vertices of $K_{3,3}$ as given in Figure 10.3.5 above, the following rotation satisfies

$r(\rho) = 3$ and hence is maximal:

$$\begin{array}{llll} 0. & 1 & 5 & 3 \\ 2. & 1 & 5 & 3 \\ 4. & 1 & 5 & 3 \\ 1. & 0 & 4 & 2 \\ 3. & 0 & 4 & 2 \\ 5. & 0 & 4 & 2 \end{array}$$

The 5 circuits of this rotation are 0 1 4 5 2 3, 0 3 4 1 2 5, and 0 5 4 3 2 1. In fact, these circuits are precisely the boundaries of the faces of the embedding of $K_{3,3}$ in the torus given in Figure 10.3.5 (above).

Note. The ideas presented in Notes 10.3.C and 10.3.D concerning the nonexistence of a circuit of length two and the requirement that circuits in a bipartite graph must be of even length can be used to prove the following (the proofs of which are to be given in Exercises 10.3.3 and 10.3.4).

Theorem 10.3.1. If every circuit induced by a rotation ρ of a graph G has length three, then ρ is a maximal rotation of G.

Theorem 10.3.2. In a bipartite graph G, if every circuit induced by a rotation ρ of G has length four, then ρ is a maximal rotation of G.

Note. Embeddings of $K_{4,4}$ and K_7 in the torus are given in Figures 10.3.6 and 10.3.7, respectively. Since $K_{3,3}$ is not planar by Theorem 10.2.3 and $K_{3,3}$ is a subgraph of $K_{4,4}$, then $K_{4,4}$ is not planar by the contrapositive of Theorem 10.2.2. Similarly K_5 is not planar by Theorem 10.2.4 and K_5 is a subgraph of K_7, then K_5 is not planar by the contrapositive of Theorem 10.2.2. Hence $\gamma(K_{4,4}) \neq 0$ and $\gamma(K_7) \neq 0$. Figures 10.3.6 and 10.3.7 then show that $\gamma(K_{4,4}) = 1$ and $\gamma(K_7) = 1$.

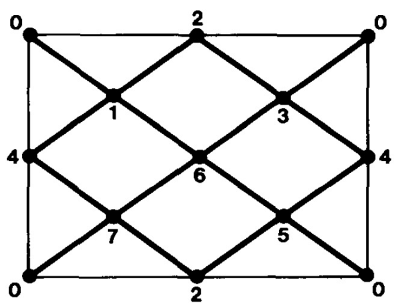

Figure 10.3.6

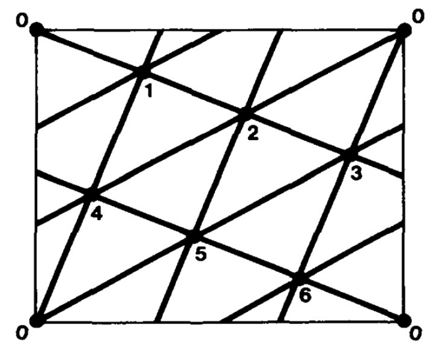

Figure 10.3.7

Next, we state "another definition" of the genus of a graph. We do not show that it is equivalent to the definition above (in terms of an embedding in a surface of genus g), but instead argue for its validity with some examples. The benefit of the new definition is that it only involves properties of the graph (in particular, the number of circuits induced by a maximal rotation) and does no involve the exploration of the topology of surfaces.

New Definition. Let G be a connected graph with p vertices, q edges, and maximal rotation ρ. The *genus of graph* G is $\gamma(G) = g$ where $p - q + r(\rho) = 2 - 2g$.

Note. By Theorem 10.1.1, $p - q + r(\rho) \leq 2$ and $p - q + r(\rho)$ is even, so g is a nonnegative integer. If G is planar if and only if $g = 0$, so that the "new definition" is consistent with the definition of planar given in Section 10.2. For the embeddings of K_5, $K_{3,3}$, $K_{4,4}$, and K_7 given in Figures 10.3.4, 10.3.5, 10.3.6, and 10.3.7, we see that each of these graphs has $p = q + r(\rho) = 0$ and so the genus of each is $g = 1$ (since we know these graphs are nonplanar, as already explained), consistent with the previous definition of genus of a graph.

Theorem 10.3.3. For the complete bipartite graph $K_{m,n}$,

$$\gamma(K_{m,n}) \geq \frac{(m-2)(n-2)}{4}.$$

Note. We now show that the genus of $K_{4,6}$ is $g = 2$. Let the vertex set be $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10\}$ (following the books notation; notice that 9 is missing from the vertex labels) and let the partite sets be $\{0, 2, 4, 6, 8, 10\}$ and $\{1, 3, 5, 7\}$. Consider the rotation with the scheme:

0.	1	3	5	7		
2.	7	5	3	1		
4.	1	3	5	7		
6.	7	5	3	1		
8.	1	3	5	7		
10.	7	5	3	1		
1.	0	2	4	6	8	10
3.	10	8	6	4	2	0
5.	0	2	4	6	8	10
7.	10	8	6	4	2	0

The reason for presenting the scheme in this form will be apparent in the proof of the corollary to Theorem 10.3.3. There are 12 circuits, each of length 4, of this rotation: 0 1 2 7, 0 3 10 1, 0 5 2 3, 0 7 10 5, 1 4 3 2, 1 6 7 4, 1 8 3 6, 1 10 7 8, 2 5 4 7, 3 4 5 6, 3 8 5 10, and 5 8 7 6. Since every circuit is of length four, then by Theorem 10.3.2, this rotation is maximal. So $p-q+r(\rho) = (4+6)-(4\cdot 6)+(12) = -2 = 2-2g$, hence $g = 2$. That is, $\gamma(K_{4,6})$. In terms of the original definition of genus of a graph, we see that this means we can embed $K_{4,6}$ in a surface of genus 2. Such a surface is called a double torus. This idea can be generalized so that we get the following corollary to Theorem 10.3.3.

Corollary 10.3.A. For the complete bipartite graph $K_{m,n}$ where m and n are both even,

$$\gamma(K_{m,n}) = \frac{(m-2)(n-2)}{4}.$$

Note. The coauthor of *Pearls in Graph Theory*, Gerhardt Ringel, gave the value of $\gamma(K_{m,n})$ for all m and n in "Das Geschlecht des vollständigen paaren Graphen," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*,**28**, 139-150 (1965).

Theorem 10.3.4. The genus of the complete bipartite graph is

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Note 10.3.E. We next consider complete graphs. The embedding of K_7 given in Figure 10.3.7 (above) is associated with the following rotation scheme (based on a clockwise rotation at each vertex):

$$\begin{array}{lcccccc} 0. & 1 & 3 & 2 & 6 & 4 & 5 \\ 1. & 2 & 4 & 3 & 0 & 5 & 6 \\ 2. & 3 & 5 & 4 & 1 & 6 & 0 \\ 3. & 4 & 6 & 5 & 2 & 0 & 1 \\ 4. & 5 & 0 & 6 & 3 & 1 & 2 \\ 5. & 6 & 1 & 0 & 4 & 2 & 3 \\ 6. & 0 & 2 & 1 & 5 & 3 & 4. \end{array}$$

Figure 10.3.7 can be used to see that the clockwise rotation produces induced circuits of length three (though we want, and need for large graphs, to use the schemes mechanically). This scheme satisfies the following rule:

Rule $\boldsymbol{\Delta}^*$. If in row i we have $\cdots, j, k, \cdots$, then in row k we have $\cdots, i, j, \cdots$.

A scheme for a rotation satisfying Rule Δ^* induces only circuits of length three. This is illustrated in Figure 10.3.9 below. From row i, we see that as we approach vertex i from vertex j, that we then proceed to vertex k (Figure 10.3.9 left). By Rule Δ^* we know by row k of the scheme that is we approach vertex k from vertex i, we then proceed to vertex j (Figure 10.3.9 second). Rule Δ^* again implies that row j reads: "$j. \cdots, k, i, \cdots$. So as we approach vertex j from vertex k, we then proceed to vertex i (Figure 10.3.9 third), completing the circuit as we see by row i again (Figure 10.3.9 right). If a scheme for a rotation satisfies Δ^*, then all circuits induced by the rotation have length 3.

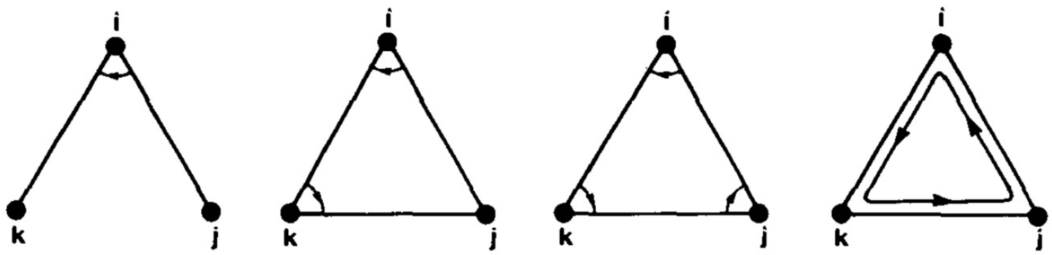

Figure 10.3.9.

Note 10.3.F. For K_8, suppose the vertex set is $\{0, 1, 2, \ldots, 7\}$. Consider the rotation:

0.	2	7	3	1	4	5	6
2.	4	1	5	3	6	7	0
4.	6	3	7	5	0	1	2
6.	0	5	1	7	2	3	4
1.	7	6	5	2	4	0	3
3.	1	0	7	4	6	2	5
5.	3	2	1	6	0	4	7
7.	5	4	3	0	2	6	1.

We find that this rotation induces 16 circuits of length three and 2 circuits of length four, so that $r(\rho) = 16 + 2 = 18$. Since the 28 edges of K_8 involve 56 edges in the circuits (in "both directions"), $56 \equiv 2 \pmod{3}$, and $18 \times 3 = 54$, then a rotation inducing 18 circuits would be maximal but it cannot consist only of length three circuits. We have in the given rotation that $16 \times 3 + 2 \times 4 = 56$ and hence this rotation is maximal. By the new definition of genus, $p - q + r(\rho) = 2 - 2g$, we have $(8) - (28) + (18) = 2 - 2g$ or $\gamma(K_8) = g = 2$. By the first definition, this means that K_8 can be embedded in a surface of genus two. Such a surface is a sphere with "two handles." This is also called a "double-torus" and can be visualized as two tori attached to each other. The fundamental domain of a double torus can be expressed as a hexagon where certain edges are joined together. In the image below, the connections (and their directions) which produce the double torus are indicated by multiple arrows on the edges of the hexagon. The embedding of K_8 is as as illustrated, where edges are given by different colors to help with identifying different ends of the same edge. Can you find the two circuits of length four?

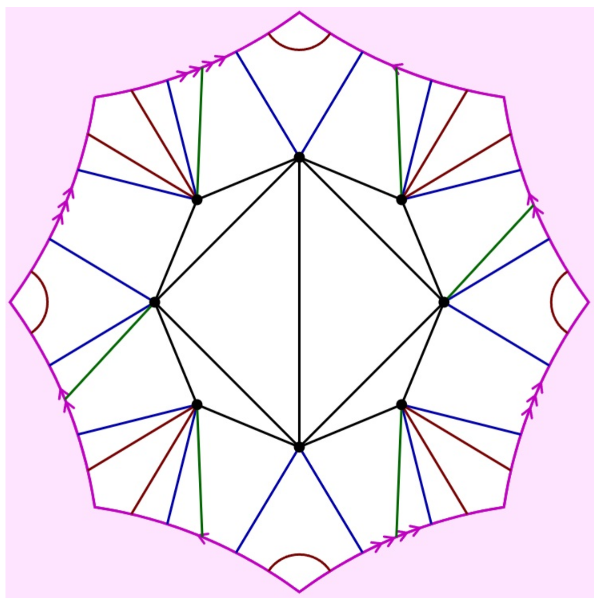

Image from the Math Stackexchange website (accessed 2/5/2023).

Theorem 10.3.5. The genus of the complete graph satisfies the inequality

$$\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}.$$

Note. A rotation of K_{12} satisfying Rule Δ^* was given by Lothar Heffter in "Ueber das Problem der Nachbargebiete [About the Problem of Neighboring Areas]," *Math-*

ematische Annalen, **38**, 477–508 (1891); available on The European Digital Mathematics Library webpage, accessed 2/6/2023. See page 494 for his rotation of K_{12}. Some biographical notes for Heffter are given in my online notes for senior/graduate level Design Theory (not an official ETSU class) on Section 1.7. Cyclic Steiner Triple Systems. Heffter proposed the problem of constructing Steiner triple systems using a cyclic automorphism in 1897 (the problem was solved by Rose Peltesohn in 1939). Since Heffter's rotation satisfies Rule Δ^*, then every induced circuit has length three and hence the rotation is maximal. In this case, the inequalities in the proof of Theorem 10.3.5 are equalities and hence $\gamma(K_{12}) = ((12) - 3)((12) - 4)/12 = 6$. Heffter's rotation is given on page 234 of the text book, though it does not resemble the version given in Heffter's 1891 paper (the vertices are different and likely permuted around from Heffter's original version). Heffter used the following rule:

Rule R*. If in row i we have $\cdots, j, k, \ell, \cdots$, then in row k we have $\cdots, \ell, i, j, \cdots$.

In Exercise 10.3.6, it is to be shown that Rule Δ^* and Rule R^* are equivalent.

Note. In Section 4 of his 1891 paper, Heffter shows that equality holds in Theorem 10.3.5 for the case of K_{12t+7}. We state this as a theorem.

Note 10.3.G. Before we consider equality in Theorem 10.3.5 for the case K_{12t+7}, we illustrate the technique for K_{19} (that is, $t = 1$). We construct row 0 using what is called a "current graph." Figure 10.3.10 gives the current graph for K_{19}, and this is circular as can be seen in Figure 10.2.11.

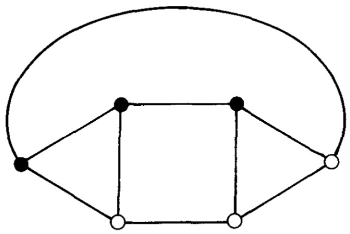

Figure 10.3.10

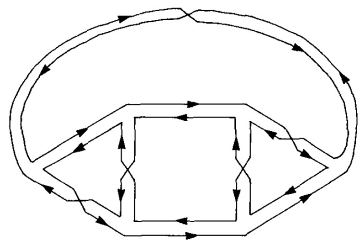

Figure 10.3.11

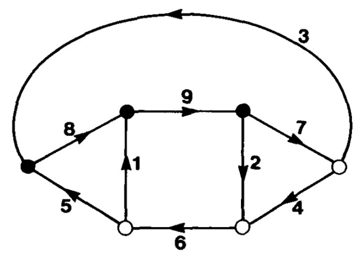

Figure 10.3.12

We label the edges of the current graph as given in Figure 10.3.12, and "log" the circuit beginning on any edge in any direction; notice that in Figure 10.3.12, the sum of the "in edges" minus the "out edges" is 0. Following the directions of Figure 10.2.11, we then generate 18 numbers associated with the edges (in each direction) where we take the edge label as positive if we traverse an edge in the direction of the arrows given in Figure 10.3.12 and as negative is we traverse an edge in the direction opposite of the arrows given in Figure 10.3.12. The log we get from Figure 10.3.12 (starting at edge 8, say) is

$$8 \quad 9 \quad 7 \quad 4 \quad -2 \quad -9 \quad -1 \quad 5 \quad -3 \quad -7 \quad 2 \quad 6 \quad 1 \quad -8 \quad -5 \quad -6 \quad -4 \quad 3$$

We then replace each negative number $-i$ with $19 - i$ to get

$$8 \quad 9 \quad 7 \quad 4 \quad 17 \quad 10 \quad 18 \quad 5 \quad 16 \quad 12 \quad 2 \quad 6 \quad 1 \quad 11 \quad 14 \quad 13 \quad 15 \quad 3$$

With this as row 0, we find row i by adding i to every entry of row 0 and reducing the result modulo 19 (such a scheme is called "additive"). This gives the rotation scheme:

0.	8	9	7	4	17	10	18	5	16	12	2	6	1	11	14	13	15	3
1.	9	10	8	5	18	11	0	6	17	13	3	7	2	12	15	14	16	4
2.	10	11	9	6	0	12	1	7	18	14	4	8	3	13	16	15	17	5
⋮																		
18.	7	8	6	3	16	9	17	4	15	11	1	5	0	10	13	12	14	2

Notice that this scheme satisfies Rule Δ^*. Suppose in row i we have $\cdots, j, k, \cdots$. Because the scheme is additive, then in row 0 we have $\cdots, j-i, k-i, \cdots$. So in the current graph there is some vertex with an edge coming in along edge $j-i$ leaves along edges $k-i$. Since the signed sum at each vertex of the current graph is 0, the third edge leaving at the vertex must be leaving with label $j-k$ (since $(j-i)-(k-i)-(j-k)=0$) and we have the configuration of Figure 10.3.13 (assuming the vertex has a clockwise rotation; we get similar behavior at a counterclockwise vertex).

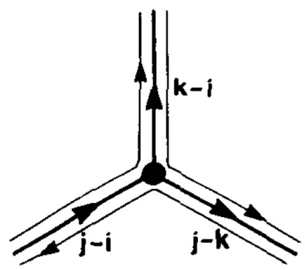

Figure 10.3.13 (slightly modified)

So elsewhere in row 0 we must have $\cdots, i-k, j-k, \cdots$. By additivity again, in row k we have $\cdots, i, j, \ldots$ and hence Rule Δ^* is satisfied. Therefore, by Note 10.3.E, all circuits induced by the rotation have length 3. Now by Theorem 10.3.1, the rotation scheme (denoted ρ) is a maximal rotation of K_{19}. Now the number of circuits is $r(\rho) = \left(2\binom{19}{2}\right)/3 = (19)(6) = 114$. So, we have $\gamma(K_{19}) = g$ where $p - q + r(\rho) = 2 - 2g$ or $(19) - (19)(18)/2 + 114 = 2 - 2g$ or $g = 20 = ((19) - 3)((19)-4)/12 = (n-3)(n-4)/12$, where $n = 19$. So we have equality in Theorem 10.3.5 when $n = 19$.

Note 10.3.H. The use of a current graph as in Note 10.3.G allows us to show that $\gamma(K_n) = \dfrac{(n-3)(n-4)}{12}$ when $n \equiv 7 \pmod{12}$. Another current graph for use in finding the genus of K_{31} is given in Figure 10.3.14. The current graph for the general case K_{12t+7} is given in Figure 10.3.15.

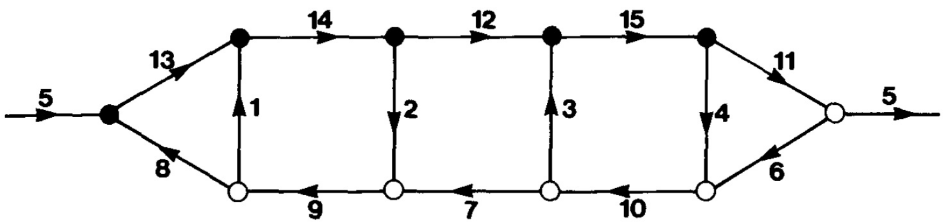

Figure 10.3.14

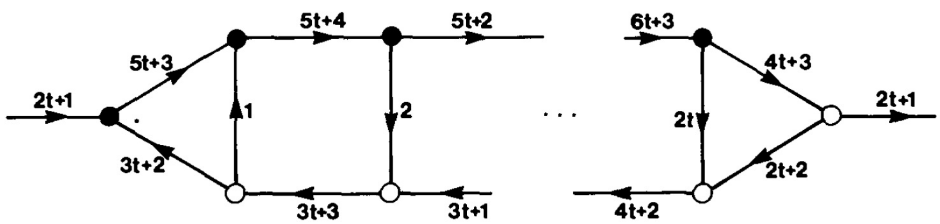

Figure 10.3.15

This justifies the following.

Theorem 10.3.A. The genus of the complete graph K_n, where $n = 12t + 7$ and $t \geq 0$, satisfies

$$\gamma(K_n) = \frac{(n-3)(n-4)}{12}.$$

Note. Notice that Theorem 10.3.A shows that Theorem 10.3.5 reduces to equality when $n \equiv 7 \pmod{12}$. In fact, Theorem 10.3.5 is best possible and reduces to equality if we round up the upper bound on $\gamma(K_n)$ (since, or course, $\gamma(K_n)$ is an integer). This is given as Theorem 10.3.6 below and is the result of a "massive endeavor" which appears in G. Ringel and J. W. Young, "Solution of the Heawood Map-Coloring Problem," *Proceedings of the National Academy of Sciences*, **60**(2), 438–445 (1968). A copy is available online on the Proceedings of the National Academy of Sciences webpage (accessed 2/6/2023). In *Pearls in Graph Theory* , contributions to the problem by Terry, Welch, and Gustin are mentioned.

Theorem 10.3.6. The genus of the complete graph K_n satisfies

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ for } n \geq 3.$$

Note. We conclude this section (and the book) with some results concerning the chromatic number of a graph. Recall that the chromatic number of graph G, $\chi(G)$, is the largest number of colors in a proper vertex coloring of G. Graph G is critical with chromatic number χ is G has chromatic number χ and every subgraph of G, other than G itself, has chromatic number less than χ.

Theorem 10.3.7. (Heawood) If G is critical, and $\gamma(G) \leq g$, where $g \geq 1$ (so that G is nonplanar), then

$$\chi(G) \leq \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$

Note. If we substitute $g = 0$ into the left side of the inequality of Theorem 10.3.7 then we see that $\chi(G)$ would be bounded by 4. This is equivalent to the Four Color Theorem (see Section 8.2. The Four Color Theorem). The case $g = 0$ is omitted from Theorem 10.3.7, due (likely) to the lengthy history of the Four Color Theorem and the fact that Heawood did not prove this case. Theorems 10.3.6 and 10.3.7 combine to give the following.

Theorem 10.3.8. (The Map Color Theorem) For every $g \geq 1$, there exists a critical graph G where $\gamma(G) \leq g$ and

$$\chi(G) = \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$

Note. In the event that $g = 1$ and we are considering a torus as the surface, Theorem 10.3.8 implies that there is a graph that can be embedded on the torus with chromatic number 7. In fact, K_7 can be embedded on the torus and of course is has chromatic number 7. See Figure 3.9(a) in my online notes for Graph Theory 2 (MATH 5450) on Section 15.1. Chromatic Numbers of Surfaces.

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