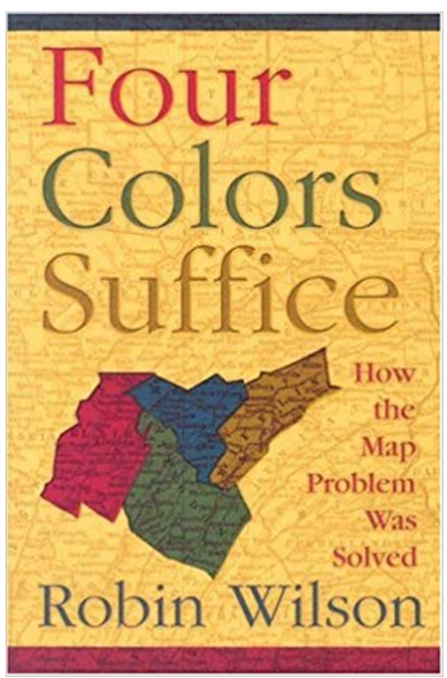
Chapter 2. Colorings of Graphs

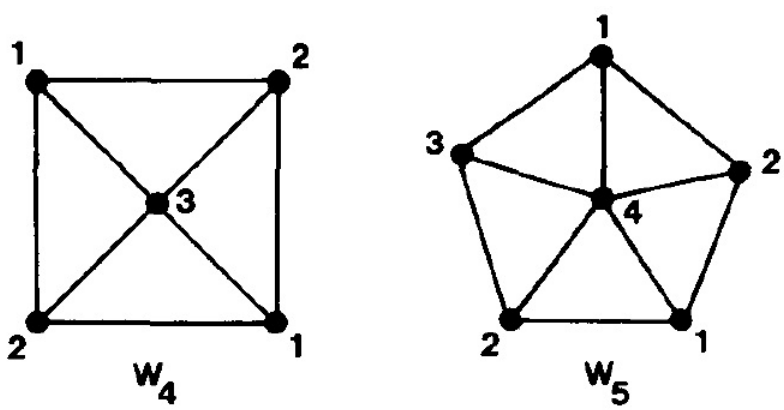
Section 2.1. Vertex Colorings

Note. One of the most influential problems in the history of graph theory is the Four Color Problem. It involves showing that for any map of countries, the countries can be assigned colors such that any two countries sharing a border have difference colors. This problem seems to first appear in 1852. After a famously false proof of 1879, a correct proof published in 1977. The 1977 proof heavily relied on computer to check cases and resulted in some controversy. A very detailed story of the problem can be found in Robin Wilson's *Four Colors Suffice: How the Map Problem was Solved*, Princeton: Princeton University Press (2002). A brief version of this history is in my online notes for Graph Theory 2 (MATH 5450) on 11.1. Colourings of Planar Maps. We will explore this in our Section 8.2, "The Four Color Theorem."



Definition. The *wheel with n spokes*, denoted W_n, is the graph that consists of an n-cycle and one additional vertex that is adjacent to all the vertices of the cycle. A *color* can be assigned to each vertex of the graph (often denoted with numbers or symbols). A *coloring* of a graph is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color.

Note. Figure 2.1.1 gives the wheels W_4 and W_5. The numerical labels can be thought of as colors and each wheel has a coloring, since adjacent vertices are given different colors.



Definition. The smallest number of colors for which G has a coloring is the *chromatic number* of G, denoted $\chi(G)$.

Note. To show that $\chi(G) = n$ we must show that G has a coloring with n colors but does not have a coloring with $n-1$ colors. Consider K_p. Certainly K_p has a coloring with p colors and no fewer colors, so $\chi(K_p) = p$. Now if we "subtract" an edge e from K_p, say the ends of e are u and v, then there is a coloring of $K_p - e$ with $p-1$ colors by assigning u and v the same color and then assigning the remaining

$p-2$ vertices each a different color (and colors different from the one assigned to vertices u and v). So the only graph on p vertices with chromatic number p is K_p. Notice that $\chi(W_4) = 3$ and $\chi(W_5) = 4$.

Note. Now consider the following graph:

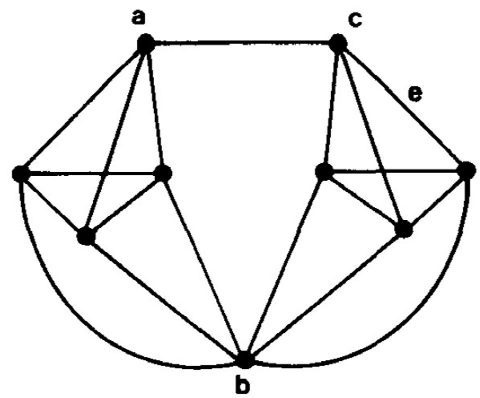
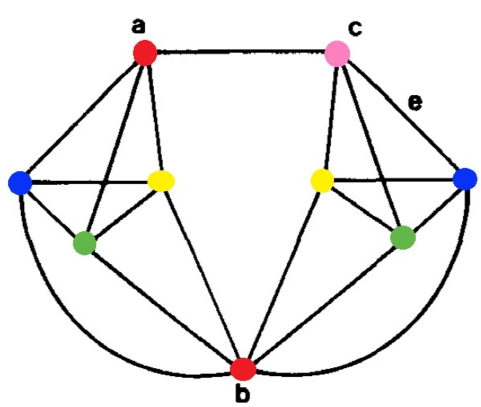


Figure 2.1.2

Since there is a subgraph isomorphic to K_4, then we see that $\chi(G) \geq 4$. *If* G can be colored with only 4 colors, then vertex a and vertex b must be the same color (because of the neighbors of b), and similarly vertex c and vertex b must be the same color. But then we have a and c of the same color, which is forbidden. So we must have $\chi(G) \geq 5$. A coloring of G with 5 colors is given below. So we have $\chi(G) = 5$.



Definition/Notation. Let G be a graph and v a vertex of G. Then "$G - v$" denotes the graph that results from G by removing vertex v and all edges incident to v. Let e be an edge of G. Then "$G - e$" denotes the graph that results from G by removing edge e (and removing no vertices of G).

Definition. If H is a subgraph of graph G and $H \neq G$, then G is a *proper subgraph* of G. If $\chi(H) < \chi(G)$ for every proper subgraph H of G, then G is a *critical* graph.

Note. A complete graph is a critical graph. Also, the graph given in Figure 2.1.2 above is a critical graph. The following is "easy to verify" and is left as Exercise 2.1.20.

Theorem 2.1.1. Every critical graph is connected.

Note. The next result guarantees, of a given graph, the existence of a sort of minimal subgraph that of G that determines the chromatic number of G.

Theorem 2.1.2. Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

Note. In fact, Theorem 2.1.2 can be easily extended (with a similar proof) to the following. Notice that in Figure 2.1.2, we have a graph G with each vertex of degree 4 and $\chi(G) = 5$, which illustrates the next result.

Theorem 2.1.3. If G is critical with chromatic number χ, then the degree of each vertex is at least $\chi - 1$.

Theorem 2.1.4. If G is a critical graph with p vertices and q edges, and G has chromatic number χ, then the relation $(\chi - 1)p \leq 2q$ holds.

Definition. The *girth* of a graph is the length of a shortest cycle in the graph.

Note. Figure 2.1.17 gives a critical graph of girth 4.

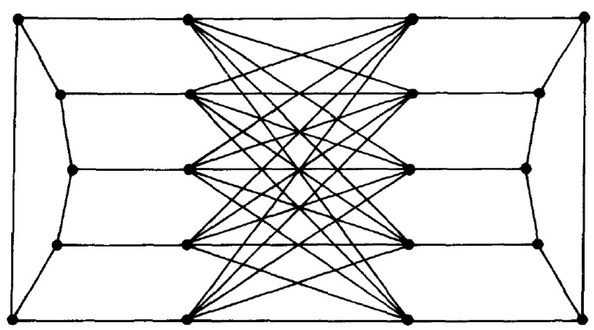


Figure 2.1.7

Note. We might think that a graph of large girth must have relatively few edges connecting vertices and hence a low chromatic number. However, the next result shows that there exist graphs with arbitrarily high girth and arbitrarily high chromatic number. This result was proved by Paul Erdős in 1961 (in "Graph Theory and Probability II," *Canadian Journal of Mathematics*, **13**, 346–352 (1961)) and also by L. Lovász in "On Chromatic Number of Finite Set-Systems," *Acta Mathematica Academiae Scientiarum Hungaricae*, **79**, 59–67 (1967)). Some details about the proof are online on Charalampos E. Tsourakakis' (of Boston University) webpage (accessed 1/21/2021).

Theorem 2.1.5. (Erdős-Lovász). For every two integers $m, n \geq 2$, there exists a graph with chromatic number n whose girth exceeds m.

Definition. A graph G is *bipartite* if $\chi(G) \leq 2$. In a coloring of bipatite graph G with two colors, the set X of all vertices of one color and the set Y of all vertices of the other color form a partitioning of the vertex set of G into two *partite* sets.

Note. In a bipartite graph G with partite sets X and Y, notice that every edge of G must have one end in X and one end in Y. This property is commonly taken as the definition of a bipartite graph.

Definition. Let G be a graph with vertices x and y. The *distance from x to y* in G, denoted $g(x, y)$, is the length of the shortest path in G from x to y. If there is not path from x to y, we say $d(x, y) = \infty$.

Note. One can show for any vertices x, y, and a in a connected graph, we have (1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$, (2) $d(x, y) = d(y, x)$, and $d(x, y) + d(y, z) \leq d(x, z)$ (this is The Triangle Inequality). These properties show that d is a *metric* on G.

Theorem 2.1.6. A graph G is bipartite if and only if every cycle in G has even length.

Note. Since a tree has no cycles, then every cycle of a tree is of even length (vacuously) and hence every tree is a bipartite graph.

Definition. The *diameter* of a graph G is the maximum distance between any two vertices of G.

Note. The diameter of a path P_n is n, the diameter of a wheel W_n is 2, and the diameter of a complete graph K_n is 1.

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