Chapter 2. Colorings of Graphs

Section 2.2. Edge Colorings

Note. In this section, we define an edge coloring, proper edge coloring, and edge chromatic number. We state (without proof) Vizing's Theorem and use it to find the edge chromatic number of complete graphs.

Definition. Two edges in a graph are *adjacent* if they are both incident with the same vertex. An *edge coloring* of a graph G is an assignment of colors to the edges of G. A *proper edge coloring* is an edge coloring with the additional property that no two adjacent edges receive the same color.

Note. Notice the different terminology between vertex colorings and edge colorings. The term "coloring" in the previous section involves vertices and includes the property that adjacent vertices were of different colors, whereas in the setting of edges the term is "proper edge coloring" when we want adjacent edges to have different colors. Figure 2.2.1 gives a proper edge coloring of a graph. Notice that the "colors" in this figure are 1, 2, 3, and 4.

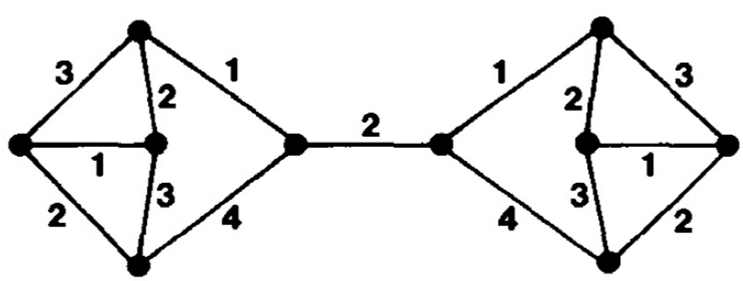


Figure 2.2.1

Theorem 2.2.1. Let G be a graph. The number of colors required for a proper edge coloring of G is greater than or equal to the maximum degree of any vertex of G.

Definition. A graph G is *regular of degree* k, or k*-regular*, if every vertex of G has degree equal to k. The *edge chromatic number* of a graph G is the minimum number of colors required for a proper edge coloring of G. A 3-regular graph with edge chromatic number 4 is a *snark*.

Note. The definition of a snark in graduate-level Graph Theory 2 (MATH 5450) is somewhat different. See my online notes for this class on Section 17.3. Snarks.

Note. The dodecahedron and the cube (see Figure 1.2.5) are 3-regular and have edge chromatic number 3. The Petersen graph (see Figure 1.1.13) is a snark and the graph in Figure 2.2.2 is an example of a "flower snark" (see Section 17.3. Snarks for more on the class of flower snarks).

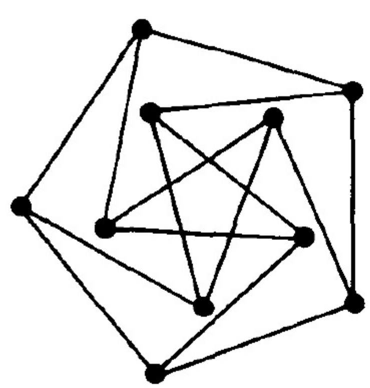


Figure 1.1.13

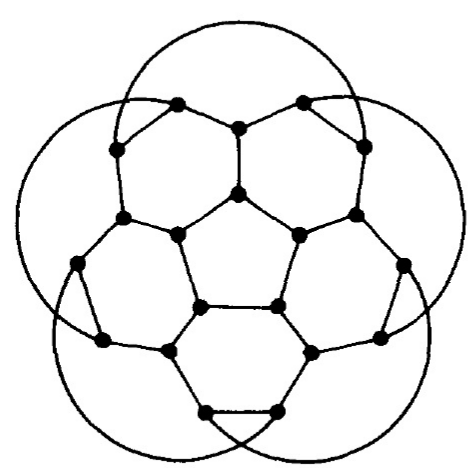


Figure 2.2.2 A snark.

Note. The best-known result on edge colorings is due to Vadim Vizing (March 25, 1937–August 23, 2017) and appeared in "On an Estimate of the Chromatic Class of a p-Graph," Diskret. Analiz. (in Russian), **3**, 25-30 (1964). It states that for a simple graph with largest vertex degree Δ, the chromatic number is either Δ or $\Delta + 1$. For a proof, see my online notes for Graph Theory 2 (MATH 5450) on Section 17.2. Vizing's Theorem. In this class, we take a special case of this result as Vizing's Theorem. we consider k-regular graphs (where, of course, the larget vertex degree is k).

Theorem 2.2.2. Vizing's Theorem.
The edge chromatic number of a k-regular graph is either k or $k + 1$.

Note. The next two results use Vizing's Theorem to give the edge chromatic number of complete graphs.

Theorem 2.2.3. The edge chromatic number of K_{2n} is $2n - 1$.

Theorem 2.2.4. The edge chromatic number of K_{2n-1} is $2n - 1$.

Definition. A spanning subgraph H of a graph G is a 1-*factor* of G if H is regular of degree 1. More generally, an r-*factor* of a graph G is an r-regular spanning subgraph.

Note. In Section 7.2. Matchings in Graphs, Scheduling Problems a *matching* in a bipartite graph is defined as a subgraph that is regular of degree 1. This idea is extended in graduate-level Graph Theory 1 (MATH 5340) where a *matching* in a general graph is defined as a set of pairwise nonadjacent edges. A *perfect matching* is defined as a spanning matching (see my online notes for Graph Theory 1 on Section 16.1. Maximum Matchings). So a 1-factor and a perfect matching are the same (though in the graduate level class the term "graph" coincides with the term "multigraph" in this class).

Note. Figure 2.2.5 gives a graph and a 1-factor of that graph.

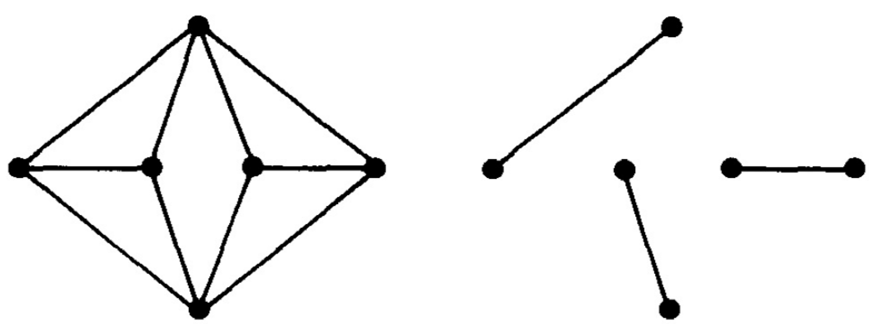
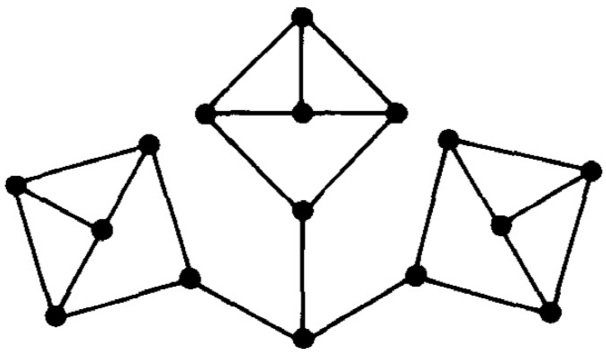


Figure 2.2.5. A 1-factor of a graph.

Notice that a graph with an odd number of vertices cannot have a 1-factor, since a 1-factor consists of a collection of isomorphic copies of K_2. But a graph with an even number of vertices also may not have a 1-factor, as illustrated in Figure 2.2.6.



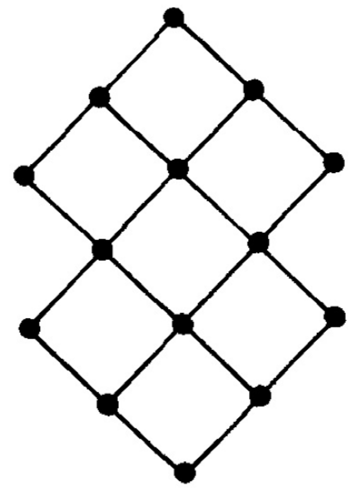


Figure 2.2.6

Revised: 11/30/2022