Chapter 2. Colorings of Graphs

Section 2.3. Decompositions and Hamilton Cycles

Note. In this section, we define graph decomposition and give several results, mostly related to cycle decompositions.

Definition. Graph G can be *decomposed* into subgraphs $H_1, H_2, \ldots, H_t$ if any two subgraphs H_i and H_j are edge disjoint, and the union of all the subgraphs H_i is G.

Note. In Figure 1.2.1, the subgraphs H_1 and H_2 of G form a decomposition of G. Subgraphs H_2 and H_3 do not form a decomposition of G since they have edges in common and their union does not include all edges of G.

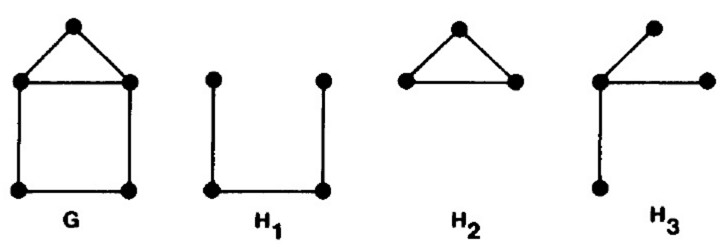

Figure 1.2.1

Definition. A *Hamilton cycle* in a graph is a cycle containing very vertex of G. A *Hamilton path* in a graph G is a path that contains every vertex of G.

Note. Notice that Figure 2.3.2 gives a decomposition of K_7 into 3 copies of a Hamilton cycle.

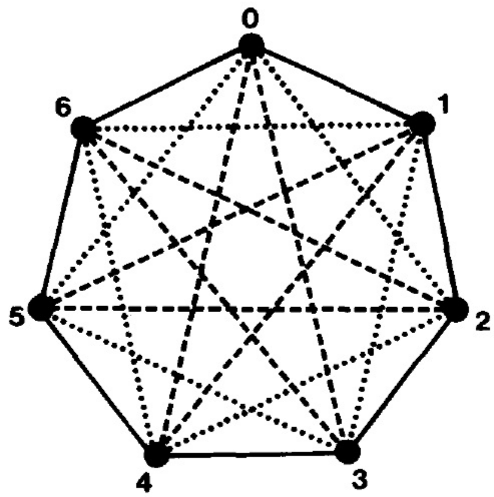

Figure 2.3.2

In fact, this result can be generalized as follows.

Theorem 2.3.1. (Lucas).
The complete graph K_{2n+1} has a decomposition into n Hamilton cycles.

Note. We can more clearly justify the proof of Theorem 2.3.1 by considering the technique of *difference methods*. This is explained in the senior/graduate class Design Theory (not an official ETSU class); see my online notes for this class on Section 1.7. Cyclic Steiner Triple Systems. It is also explored in graduate level Graph Theory 1 (MATH 5340) in Supplement. Graph Decompositions: Triple Systems.

Note. We now give some results concerning various decompositions.

Theorem 2.3.2. K_{2n} has a decomposition into $n-1$ Hamilton cycles and a 1-factor.

Theorem 2.3.3. K_{2n} has a decomposition into n Hamilton paths.

Theorem 2.3.4. The complete graph K_{2n} has a decomposition into $2n-1$ paths consisting of one path of each length k for $k = 1, 2, 3, \ldots, 2n-1$.

Note. Graph decompositions are a vibrant area of research. One particular open problem is due to A. Gyárfás and J. Lehel who in 1976 conjectured: "Given $T_1, T_2, \ldots, T_{2n-1}$, where each T_i is a tree with i edges, then K_{2n} can be decomposed into these given trees."

Note. Recall (from the previous section) that a 3-regular graph with edge chromatic number 4 is a *snark*.

Theorem 2.3.5. A snark has no Hamilton cycle.

Note. The converse of Theorem 2.3.5 does not hold. That is, a graph may have no Hamilton cycle but not be a snark. In Figure 2.3.4 is a graph that has no Hamilton cycle, is cubic, and is has edge chromatic number 3 (recall that a snark is a cubic graph with edge chromatic number 4).

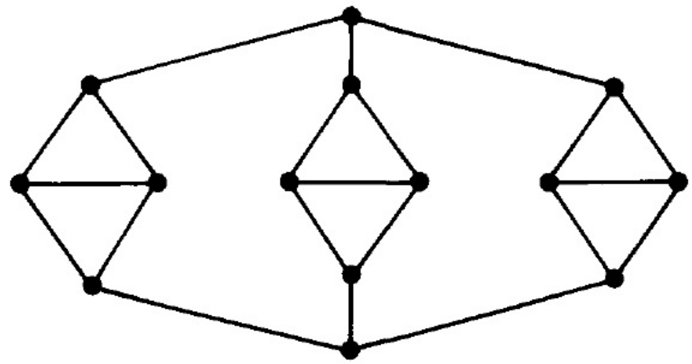

Figure 2.3.4