Chapter 2. Colorings of Graphs

Section 2.4. More Decompositions

Note. In this section, we address additional graph decompositions and define a "bridge" in a graph.

Note. We can use the "turning trick" (as Hartsfield and Ringel call it) to perform certain decompositions. This is more commonly thought of as using a particular type of permutation. In Figure 2.4.1, we have a shaded P_3 in the Petersen graph. If the P_3 is rotated around through 1/5 of a turn then the shaded P_3 produces a new P_3 in the Petersen graph. Continuing to rotate, we see that the P_3 produces all edges of the Petersen graph and so the Petersen graph can be decomposed into 5 copies of P_3. Similarly, in Figure 2.4.2 the cube Q_3 can be decomposed into 4 copies o fP_3.

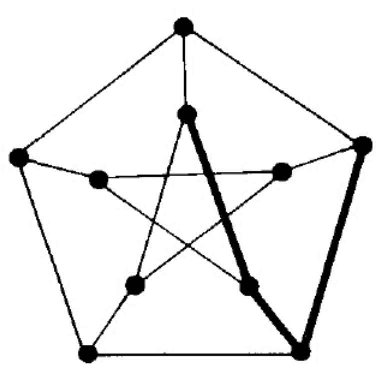

Figure 2.4.1

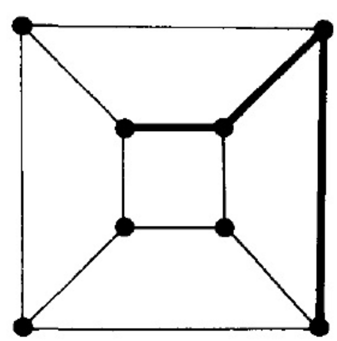

Figure 2.4.2

Theorem 2.4.A. Let G be a cubic graph. Then G has no decomposition into subgraphs, each of which is isomorphic to a path of length four.

Definition. A *bridge* in a connected graph G is an edge of G whose removal disconnects G into two subgraphs. The two subgraphs are the *banks* of the bridge. In a disconnected graph, the subgraphs that are connected and not contained in any larger connected subgraph are the *components* (or *connected components*); that is, the components are the maximal connected subgraphs.

Note. A bridge is also sometimes called a *cut edge* (though the term "bridge" is probably more common). See my online notes for graduate-level Graph Theory 1 (MATH 5340) on Section 3.2. Cut Edges.

Note. The next result is to be proved in Exercise 2.4.22.

Theorem 2.4.1. Let G be a connected graph. Then G is a tree if and only if every edge of G is a bridge.

Note. Trivially, every graph has a decomposition into paths of length one. The next result classifies those connected graphs with decompositions into paths of length two.

Theorem 2.4.2. A connected graph is decomposable into subgraphs each isomorphic to a path of length two if and only if the graph has an even number of edges.

Note. For some detailed information on graph decompositions involving decomposing complete graphs into graphs on three vertices, see my online notes for Graph Theory 1 (MATH 5340) on Graph Decompositions: Triple Systems.

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