Chapter 3. Circuits and Cycles

Section 3.1. Eulerian Circuits

Note. The earliest known paper on graph theory is by Leonhard Euler: "Solutio problematis ad geometriam situs pertinentis" (Solution to the geometry of position), *Comment. Academiae Sci. I. Petropolitanae* **8** (1736), 128–140.

Leonhard Euler (April 15, 1707–September 18, 1783)

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The paper addressed the Kónigsberg Bridge Problem. The Pregel River runs through Königsberg, formerly in Germany but now in Russia and called Kaliningrad (see the brief WolframMathWorld article on the problem). A diagram of the river and the associated graph appears in Figure 3.1.1.

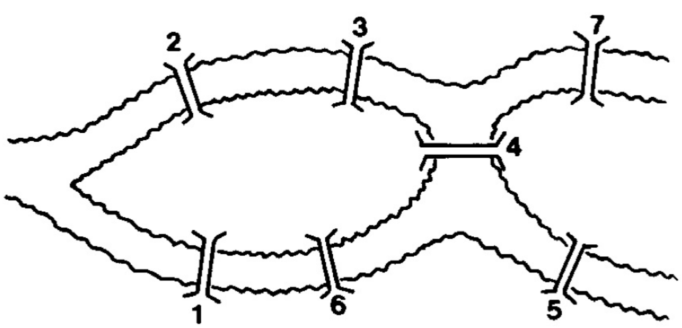

Figure 3.1.1. The Königsberg Bridges

The problem is to find a trajectory to walk across the bridges such that each bridge is crossed just once. By representing the land areas as vertices and the bridges as edges of a multigraph, as given in Figure 3.1.2, the question can be translated into graph theoretic terms concerning the existence of a "walk" through the graph with certain properties. This inspires the definition of walk given below.

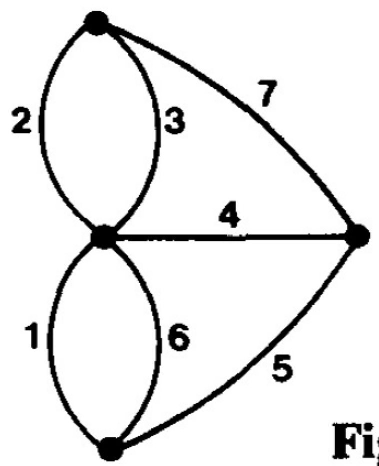

Figure 3.1.2

Note. As we see in Figure 3.1.1, the Kónigsberg Bridge Problem requires us to consider multigraphs. Since a pair of vertices does not uniquely define an edge in a multigraph, then we must label both vertices and edges when considering multigraphs. By convention, we represent vertices by capital letters and edges by lower-case letters (in this section).

Definition. A *walk* in a pseudograph G is an alternating sequence

$$A_1 e_1 A_2 e_2 A_3 \cdots A_{n-1} e_{n-1} A_n$$

of vertices and edges of G, with the property that every edge e_i is incident with the vertices A_i and A_{i+1}, and $A_i \neq A_{i+1}$ if e_i is not a loop. If $A_1 \neq A_n$ then the walk is an *open walk* and if $A_1 = A_n$ then it is a *closed walk*.

Definition. A *trail* in a pseudograph G is a walk in G with the property that no edge is repeated. A *path in a pseudograph* G is a trail in G with the property that no vertex is repeated.

Definition. The *length* of a walk is the number of edges in the walk. A *closed trail* (or *circuit*) is a trial whose endpoints are the same vertex. A *closed path* (or *cycle*) is a path whose endpoints are the same vertex. A cycle of length 3 is a *triangle*. In a pseudograph, a *loop* is a cycle of length 1, and a cycle of length 2 is a *lune*. The *degree* of a vertex V in a pseudograph is the number of edges incident with V, counting loops twice. An *Eulerian circuit* in a pseudograph G is a circuit that contains every edge of G.

Example. Consider the pseudograph in Figure 3.1.3.

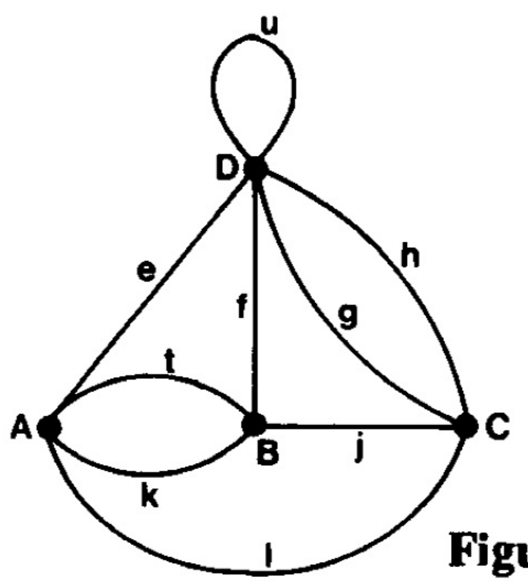

Figure 3.1.3

A walk from vertex A to vertex C is $AtBfDuDfBjC$; this walk is of length 5. Since edge f is repeated, this is not a trail, but a trail from A to C is $AtBfDhCgDeAlC$.

Since vertices A, C, and D are repeated, this is not a path. A path from A to C is $AtBfDhC$. A triangle in the graph is the cycle $BjCgDfB$. Vertex D has degree 6 (since loop u contributes twice to the degree of D). A Eulerian circuit is

$$AeDuDhCgDfBjClAkBtA.$$

Note. The Kónigsberg Bridge Problem amounts to finding an Eulerian circuit in the graph of Figure 3.1.2. While exploring this problem, Euler proved the following (which shows that there is no solution to the Kónigsberg Bridge Problem).

Theorem 3.1.1. Euler's Theorem.
If a pseudograph G has an Eulerian circuit, then G is connected and the degree of every vertex is even.

Note. In fact, the converse of Euler's Theorem also holds. An argument for it was given by Carl Hierholzer (October 2, 1840–September 13, 1871). He discussed it with his colleagues, but died unexpectedly shortly afterwards. One of them arranged for the result to be published and it appeared in: Carl Hierholzer and Chr. Wiener, "Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren [On the Possibility of Traversing a Line-System without Repetition or Discontinuity]," *Mathematische Annalen*, **6**, 30-32 (1873). This brief work is reprinted in English in N. L. Biggs, E. K. Loyd, and R. J. Wilson's *Graph Theory: 1736-1936* (Oxford: Oxford University Press, 1976); see pages 11 and 12. The proof of this result is today called "Hierholzer's algorithm."

Theorem 3.1.2. Hierholzer's Theorem.
If a pseudograph G is connected and the degree of every vertex of G is even, then G has an Eulerian circuit.

Note. We need a preliminary result for giving a proof of Theorem 3.1.2.

Lemma 3.1.3. If every vertex in a pseudograph G has positive even degree, then any given vertex of G lies on some circuit of G.

Note. We are now ready to present a proof of Hierholzer's Theorem (Theorem 3.1.2).

Note. In Graph Theory 1 (MATH 5340), "Fleury's Algorithm" (originally presented in 1883) is stated and proved. It gives a construction of an Eulerian circuit in a an pseudograph with all vertices of even degree. See my online notes for Graph Theory 1 on Section 3.3. Euler Tours.

Note. Recall from Section 2.2. Edge Colorings that a 2-factor of a graph is a spanning subgraph that is regular of degree 2. So a 2-factor consists of one or more cycles (one if the 2-factor is connected). We take the same definition in the setting of pseudographs. In Figure 3.1.5 below, the octahedron graph is decomposed into two 2-factors in two different ways (on the left using dark edges and regularly represented edges, and on the right using dotted edges and regularly represented edges). The fact that such decompositions exist is not a coincidence, as shown in the following theorem.

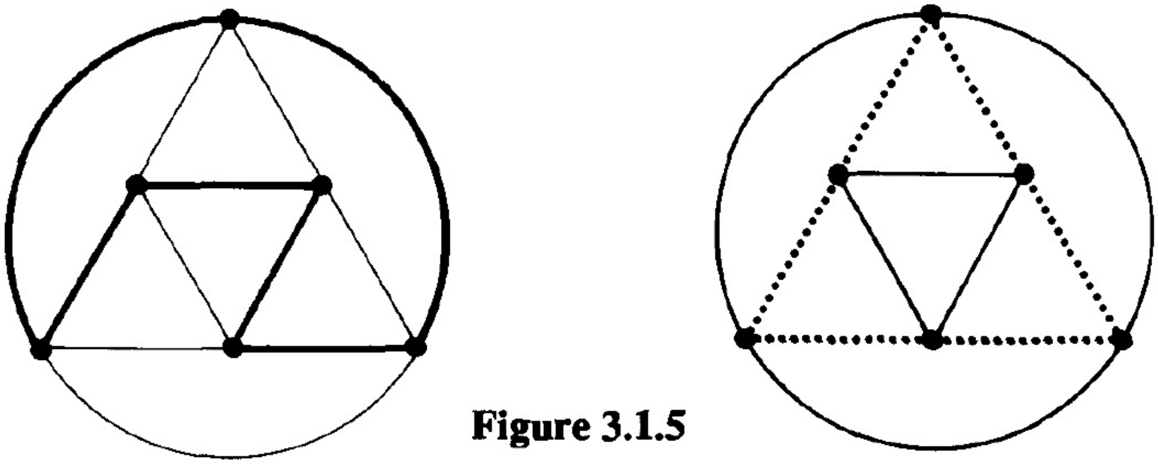

Figure 3.1.5

Theorem 3.1.4. If a pseudograph G is regular of degree 4, then G has a decomposition into two 2-factors.

Note. The previous theorem can be generalized to pseudographs for which all vertices are of even degree (and the 2-factors are replaced with a collection of cycles). This result is due to O. Veblen (in 1912/13); see my online notes for Graph Theory 1 (MATH 5340) on Section 2.4. Decompositions and Coverings.

Theorem 3.1.5. Veblen's Theorem.
A pseudograph G has a decomposition into cycles if and only if every vertex of G has even degree.

Note. In an Eulerian circuit, we require that we "start" and "end" at the same vertex and that we traverse all the edges of a graph. We see from Euler's Theorem (Theorem 3.1.1) that this requires the graph to have very vertex of even degree. If we consider traversing all the edges of a graph but do not require that we "start" and "end" at the same vertex, we can also make some progress concerning graphs with some vertices of odd degree. This motivates the following definition and theorem.

Note. An *Eulerian trail* is a pseudograph G is a trail that contains every edge and every vertex of G.

Theorem 3.1.6. A pseudograph G has an Eulerian trail if and only if G is connected and has precisely two vertices of odd degree.

Note. Hartsfield and Ringel describe applications of Theorem 3.1.6. They consider the "ornament" of Figure 3.1.7 below and say that an artist can construct it from a single piece of wire without cutting by starting at an odd degree vertex and ending at an odd degree vertex (the two odd degree vertices are of degree 3 and are the left-most and right-most vertices in Figure 3.1.7).

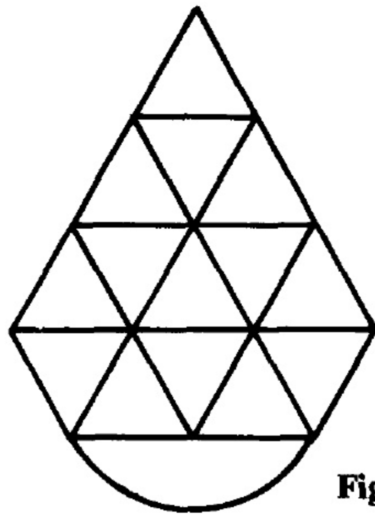

Figure 3.1.7

A related application is a "tracing puzzle." The game is to trace out a particular figure without picking up your pencil and without repeating lines. The educational website Transum.org has a webpage illustrating these tracing puzzles (accessed 3/13/2021).

Note. We see from the last theorem that vertices of odd degree play a key role in trail decompositions of a graph. We generalize Theorem 3.1.6 as follows.

Theorem 3.1.7. Listing's Theorem.
If G is a connected pseudograph with precisely $2h$ vertices of odd degree, $h \neq 0$, then there exists h trails in G such that each edge of G is exactly one of these trails. Furthermore, fewer than h trails with this property cannot be found.

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