Chapter 3. Circuits and Cycles

Section 3.2. The Oberwolfach Problem

Note. The town of Oberwolfach is in Baden-Württemberg in southwestern Germany. It is the home of the Oberwolfach Research Institute for Mathematics or *Mathematisches Forschungsinstitut Oberwolfach*. When meals were served during meetings, the institute tried to have everyone sit next to everyone else at least once. Most of the tables have six places, but some tables are larger than others. Idealized, the problem is, given $2n + 1 = t_1 + t_2 + \cdots + t_s$ and s round tables $T_1, T_2, \ldots, T_s$ such that table T_i seats t_i people, $2n + 1$ people should have dinner for n nights so that after a meeting everyone has sit next to everyone else. Gerhard Ringel (one of the authors of our text book) posed this problem and named it the "Oberwolfach Problem" in 1967. Ths is documented in R. K. Guy's "Unsolved Combinatorial Problems" in *Combinatorial Mathematics and its Applications*, ed. D. Welsh, Academic Press (1971), pp. 121–127.It remains, in its general version, unsolved, but many special cases have been solved.

Note. The Oberwolfach Problem can be stated in graph theoretical terms as follows.

> **Oberwolfach Problem.** Given a graph T with $2n+1$ vertices that is regular of degree two (meaning that T consists of one or more cycles of various lengths), decompose the complete graph K_{2n+1} into n subgraphs isomorphic to T.

Graph T represents the collection of tables where the vertices are the people, so that adjacency in T corresponds to two people sitting next to each other. If there is only one table, then the graph T is a Hamilton cycle in K_{2n+1}. A solution is then given by a Hamilton cycle decomposition of K_{2n+1}, which we have from Theorem 2.3.1.

Note. We denote graph T, which represents s tables $T_1, T_2, \ldots, T_2$ where table T_i sets t_i people, as $(t_1, t_2, \ldots, t_s)$ (by convention, we take $t_i \leq t_{i+1}$ for $1 \leq i \leq s$). Figure 3.2.1 gives a solution to the Oberwolfach Problem for graph T as $(4, 3)$ (so that $2n + 1 = 7$).

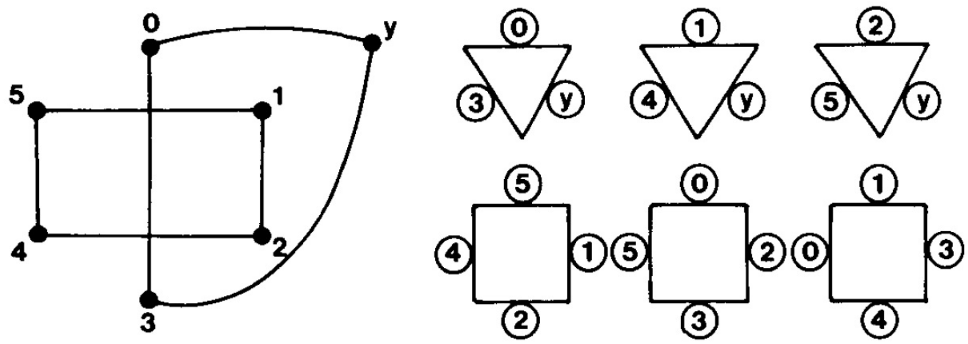

Figure 3.2.1

The "turning trick" is to be applied to the 3-cycle and 4-cycle given on the left to produce the three seatings given on the right. That is, we take the 3-cycle $(y, 0, 3)$ and the 4-cycle $(1, 2, 4, 5)$ (where we represent an n-cycle on vertices $v_0, v_1, \ldots, v_{n-1}$, with v_i and v_j adjacent if and only if $j \equiv i + 1 \pmod{n}$, as $(v_0, v_1, \ldots, v_{n-1})$ or any cyclic permutation of this) and apply the permutation $(y)(1, 2, 4, 5)$ to get the seatings:

$$(y, 0, 3), (1, 2, 4, 5); \quad (y, 1, 4), (2, 3, 5, 0); \quad (y, 2, 5), (3, 4, 0, 1)$$

(notice that applying the permutation again, that is "turning" again, returns the last seating to the first). Additional solutions are given in Figures 3.2.2 and 3.2.3.

Note. For the remainder this section, we concentrate of regular graphs, mostly 3-regular graphs. Recall that a *bridge* in a connected graph G is an edge whose removal disconnects G.

Theorem 3.2.1. A regular graph of even degree has no bridge.

Note. Our next three results concern cubic graphs and decompositions into 1-factors, 2-factors, and paths of length two.

Theorem 3.2.2. A cubic graph that contains a bridge is not decomposable into three 1-factors.

Note. Recall from Section 2.2. Edge Colorings that a snark is a cubic graph with edge chromatic number four (also recall that a color class in an edge coloring forms a 1-factor). So a snark is not decomposable into three 1-factors. The following decomposition result is due to Julius Petersen (of the Petersen graph fame) and appeared in "Die Theorie der regulären Graphen [The Theory of Regular Graphs]," *Acta Mathematica*, **15**, 193–220 (1891).

Theorem 3.2.3. (Petersen) A cubic bridgeless graph G has a decomposition into a 1-factor and a 2-factor.

Note. We do not present a proof of Petersen's Theorem (Theorem 3.2.3), but a proof can be found in my Graph Theory 2 (MATH 5450) notes on Section 16.4. Perfect Matchings and Factors (see Theorem 16.14). There, the result is stated as: "Every 3-regular graph without cut edges has a perfect matching." This is equivalent to our statement, since 3-regular and cubic mean the same thing, a "cut edge" is a bridge, and a perfect matching is a 1-factor; if we remove the edges of a 1-factor from a cubic graph, what is left must be a collection of edges forming a 2-regular spanning subgraph (that is, a 2-factor). In fact, since since in a cubic graph the set of edges of a 1-factor has as its complement with respect to the edge of the graph as a set of edges of a 2-factor, we could also state Petersen's Theorem as: "Every cubic bridgeless graph has a 2-factor." An application of Petersen's Theorem (Theorem 3.2.3) is the following.

Theorem 3.2.4. Every cubic bridgeless graph is decomposable into paths of length three.

Note. A "famous conjecture" of Claude Berge proved by Limin Zhang in "Every 4-regular Simple Graph Contains a 3-regular Subgraph," *Journal of Changsha Railway Institute*, No. 1, 130–154 (1985) is the following.

Theorem 3.2.5 (Berge and Zhang) Every 4-regular graph contains a 3-regular subgraph.

Note. Figures 3.2.7 and 3.2.8 on page 64 of the text book give examples of (small) 4-regular graphs which have 1-factors (and so have 3-regular subgraphs, as insured by Theorem 3.2.5).

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