Chapter 3. Circuits and Cycles

Section 3.3. Infinite Lattice Graphs

Note. This is the only section in the text book which deals with infinite graphs. We give several definitions in this brief section, but state no theorems.

Definition. The infinite graph L_2 has as its vertex set all points in the Cartesian plane whose coordinates are both integers, and as its edge set all edges joining pairs of vertices which are a geometric distance one apart.

Note. Of course L_2 has an infinite number of vertices and an infinite number of edges. In Graph Theory 1 (MATH 5340), infinite graph L_2 is called the "square lattice"; see my online note for Graph Theory 1 on Section 1.6. Infinite Graphs (notice Figure 1.27). The name L_2 is based on the fact that it is a lattice in the two-dimensional plane. Figure 3.3.1 gives a portion of L_2.

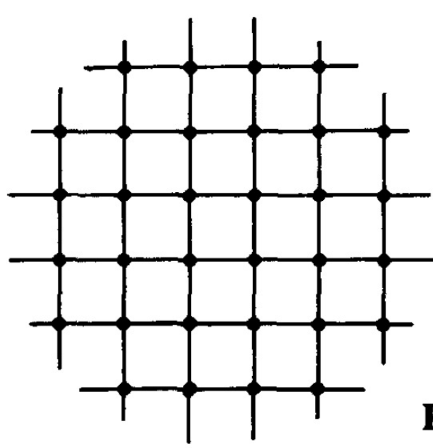

Figure 3.3.1

We now give some definitions which extend some of the ideas for finite graphs to infinite graphs and illstrate them in L_2.

Definition. A *Hamilton line* in an infinite graph is a connected 2-factor.

Note. Notice that a Hamilton line in L_2 is similar to both the idea of a Hamilton cycle in a finite graph (though in an infinite graph there is no way to "return to the stating position" and to visit all vertices). Two patterns that show the existence of a Hamilton line in L_2 are given in Figures 3.3.2 and 3.3.3.

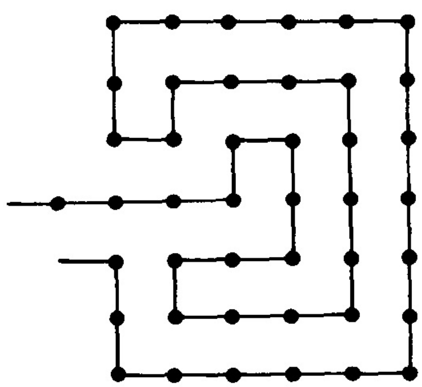

Figure 3.3.2

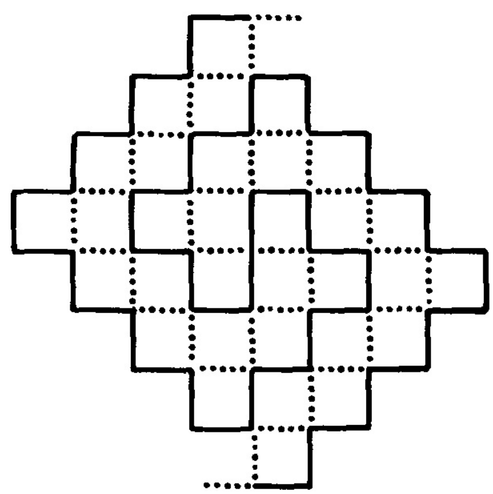

Figure 3.3.3

Definition. An *Eulerian line* in an infinite graph is an Eulerian trail that is infinite in both directions. That is, it is an infinite walk in the graph such that every edge of the graph is present in the walk exactly one.

Note. Figures 3.3.4 and 3.3.5 give two patterns that show the existence of an Eulerian line in L_2. Notice that we can rotate the Eulerian line of Figure 3.3.5 through 180° and it remains unchanged (showing the existence of a symmetry).

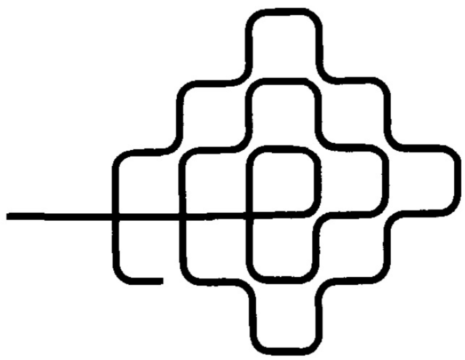

Figure 3.3.4

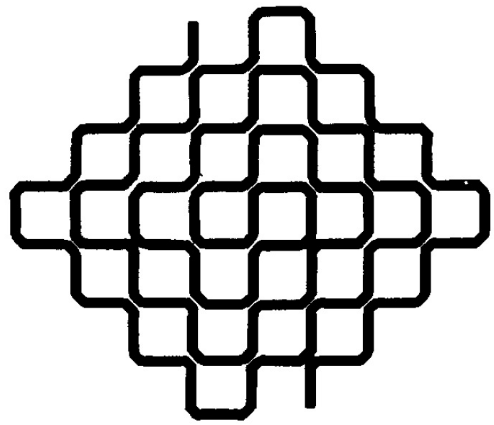

Figure 3.3.5

Definition. A trail in an infinite graph that begins at one vertex y, has infinite length, and includes every edge of the graph exactly once, is a *one-way Eulerian trail*.

Note. As in finite graph with Euler trails, it is necessary for an infinite graph with a one-way Eulerian trail that vertex y is of odd degree and all other vertices are of even degree. However, for infinite graphs this is not also sufficient for the existence of a one-way Eulerian trail (as to be shown be example in Exercise 3.3.17).

Definition. A *one-way Hamilton path* in an infinite graph is a path (so the edges are distinct) of infinite length, starting at a vertex z and passing through every vertex of the graph.

Note. The degree of vertex z has no impact on the existence of a one-way Hamilton path in an infinite graph. Two more examples of infinite graphs, other than L_2 are given in Figures 3.3.6 (the "infinite spiderweb graph") and 3.3.7 (the "triangluar lattice").

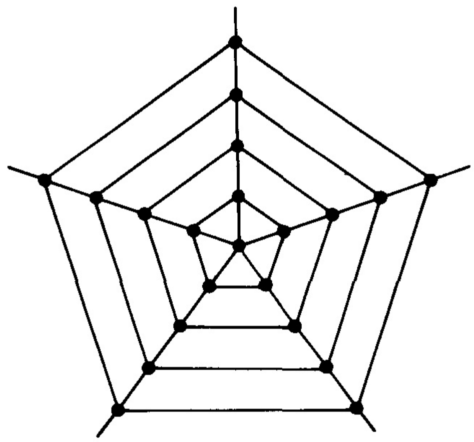

Figure 3.3.6

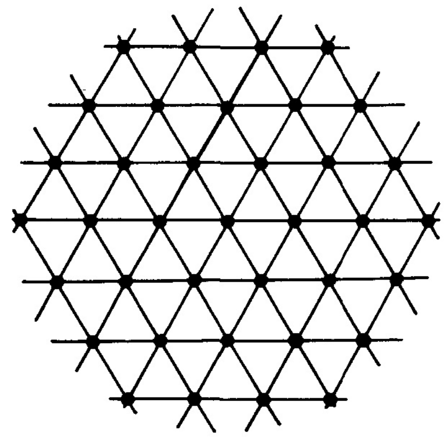

Figure 3.3.7