Chapter 4. Extremal Problems

Note. Extremal problems in graph theory involve finding a largest or smallest graph with a given property. These measures of size involve the number of vertices and/or the number of edges, leading us to the following definitions.

Definition. The number of vertices in a graph G is the *order* of G. The number of edges in G is the *size* of G.

Note. In Section 4.1 we consider the largest graphs with a given chromatic number, and classify the largest graphs that to not have complete subgraphs (in Turan's Theorem, Theorem 4.1.2). In Section 4.2 we define a g-cage as a smallest graph of girth g, and give some examples of such graphs. In Section 4.3 we consider edge colorings of complete graphs with two colors and state a result concerning monochromatic complete subgraphs (Ramsey's Theorem, Theorem 4.3.1).

Section 4.1. A Theorem of Turan

Note. In this section we present a classical result in extremal graph theory. We consider the largest graphs (that is, those with the most edges) with certain chromatic numbers (in Theorems 4.1.A and 4.1.1) and with subgraphs that are not complete (in Theorems 4.1.2* and 4.1.2, Turan's Theorem).

Definition. An *induced subgraph* of a graph G is a subgraph of G obtained by taking a subset W of the vertices of G together with every edge of G that has both endpoints in W.

Note. Recall that the greatest integer function is denoted $\lfloor x \rfloor$ and gives the largest integer less than or equal to x. You might think of it as the "rounding down" function, though it is often called the *floor symbol*. The least integer function is denoted $\lceil x \rceil$ and gives the least integer greater than or equal to x. You might think of it as the "rounding up" function, though it is often called the *ceiling symbol*. For example, notice that $\lfloor 11/2 \rfloor = 5$ and $\lceil 11/2 \rceil = 6$. We use these functions to prove a special case of our Theorem 4.1.1.

Theorem 4.1.A. The largest graph G (that is, with the most edges) with chromatic number two and n vertices is a complete bipartite graph K_{n_1,n_2} where $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.

Note. In Theorem 4.1.A, we have $m_2 = \lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor$. In fact, this is a special case with $k = 2$ of a more general idea:

$$n = \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n+1}{k} \right\rfloor + \left\lfloor \frac{n+2}{k} \right\rfloor + \cdots + \left\lfloor \frac{n+k=1}{k} \right\rfloor.$$

Notice that any two terms on the right-hand side of this equation differ by at most 1; that is, we have $n = n_1 + n_2 + \cdots + n_k$ (where we take $n_i = \lfloor (n+i-1)/2 \rfloor$ for $i = 1, 2, \ldots, k$) and $|n_i - n_j| \le 1$ for $1 \le i, j \le k$. We employ this notation in the next theorem, which reduces to Theorem 4.1.A when $k = 2$.

Theorem 4.1.1. The largest graph G (that is, with the most edges) with chromatic number k and n vertices is a complete k-partite graph $K_{n_1,n_2,\ldots n_k}$ where $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \leq 1$.

Note. We now turn to Turan's Theorem. It has the same conclusion as Theorem 4.1.1, but it has a weaker hypothesis. It is stated in terms the nonexistence of complete subgraphs of a certain size, as opposed to vertex colorings and the chromatic number.

Theorem 4.1.2. Turan's Theorem.
The largest graph (that is, with the most edges) with n vertices that contains no subgraph isomorphic to K_{k+1} is a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \leq 1$.

Note. Notice that if a graph is k-chromatic, then it cannot have a subgraph isomorphic to K_{k+1}, so that the hypotheses of Theorem 4.1.1 imply the hypotheses of Turan's Theorem (Theorem 4.1.2). However, the hypotheses of Turan's Theorem do not imply the hypotheses of Theorem 4.1.1 because, for example, a cycle of length five contains no K_3 (so it satisfies the hypotheses of Turan's Theorem with $k = 2$), but its chromatic number is $3 \neq 2 = k$. First, we prove a special case of Turan's Theorem with $k = 2$.

Theorem 4.1.2*. The largest graph (that is, with the most edges) with n vertices that contains no triangle is the complete bipartite graph K_{n_1,n_2} with $n = n_1 + n_2$ and $|n_1 - n_2| \leq 1$.

Note. Before we prove Turan's Theorem (Theorem 4.1.2), we need a preliminary lemma.

Lemma 4.1.3. If G is a graph on n vertices that contains no K_{k+1} then there is a k-partite graph H with the same vertex set as G such that $\deg_G(z) \leq \deg_H(z)$ for every vertex z of G.

Note. We now have the equipment to give a quick proof of Turan's Theorem (Theorem 4.1.2).

Definition. A complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \leq 1$ is called a *Turan graph*, denoted $T_{n,k}$.

Note. Of course, we can reword Turan's Theorem as: "The largest graph (that is, with the most edges) with n vertices that contains no subgraph isomorphic to K_{k+1} is the Turan graph $T_{n,k}$." A drawing of the Turan graph $T_{15,4}$ (where $n_1 + n_2 + n_3 + n_4 = 3 + 4 + 4 + 4 = 15$) is given in Figure 4.1.1 below.

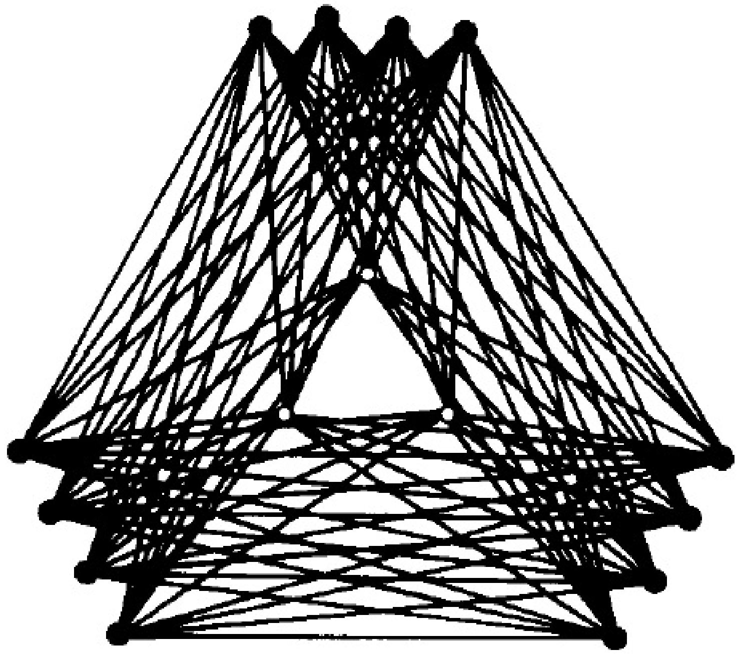
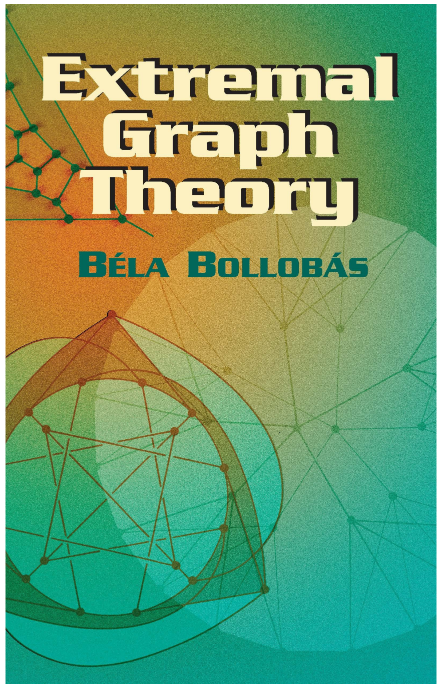


Figure 4.1.1. The Turan graph $T_{15,4}$.

Note. Turan's Theorem was first given by Paul Turán in "Eine Extremalaufgabe aus der Graphentheorie" [in Hungarian; "On an extremal problem in graph theory"] *Matematikai és Fizikai Lapok* **48**, 436–452 (1941). This result is addressed in the graduate class Graph Theory 2 (MATH 5450), where it is also applied to a result in combinatorial geometry. See my online notes for Graph Theory 2 on Section 12.2. Turan's Theorem.

Note. More generally, a classic work on the topic of this chapter is Bela Bollabas' *Extremal Graph Theory*, Academic Press (1978); it has been in print by Dover Publications since 2004.



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