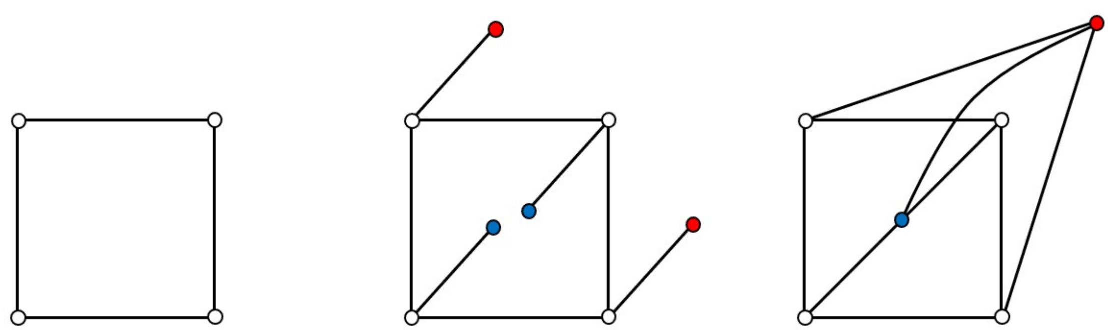
Section 4.2. Cages

Note. Recall from Section 2.1. Vertex Colorings that the *girth* of a graph is the length of a shortest cycle in the graph. In this brief section we consider the smallest cubic graph (in terms of the number of edges) of a given girth.

Definition. A smallest (in terms of the number of edges) cubic graph with given girth g is called a g-*cage*.

Note. If $g = 3$ then rather clearly the 3-cage is K_4 (it must contain a 3-cycle and in order for each vertex to be of degree three, three new edges must be added and these are then incident to a new fourth vertex). If $g = 4$, then we can start with a 4-cycle and again see that we must add another edge to vertex (and these edges must be distinct, or we would introduce a 3-cycle, violating the girth condition). We can have pairs of these new edges incident to a single additional vertex each, provided paired edges are incident to vertices opposite each other in the 4-cycle. But then these two new vertices are only degree two, so we need to add a new edge joining each of these. This produces a graph isomorphic to $K_{3,3}$ (see the figure). "In fact" (Hartsfield and Ringel state on page 77) these are the unique 3-cage and 4-cage, respectively.



Note. We now classify the 5-cage and 6-cage. The proofs of these two results are ad hoc, but such arguments are common in discrete math when dealing with small cases.

Theorem 4.2.1. The Petersen graph is the unique 5-cage.

Theorem 4.2.2. The Heawood graph of Figure 4.2.4 is the unique 6-cage.

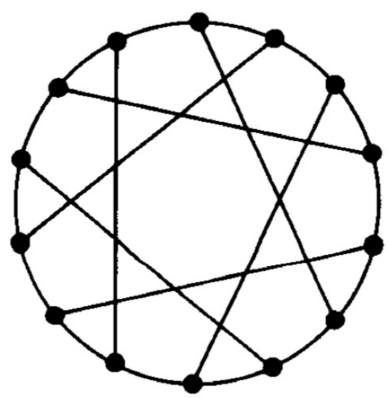


Figure 4.2.4. The Heawood graph (1890).

Note. The unique 7-cage is the McGee graph of Figure 4.2.6. The unique 8-cage is the Tutte-Coxeter graph in Figure 4.27. The 9-cage is not known. There are three distinct 10-cages, of of which is given in Figure 4.2.8. The 11-cage has 112 vertices and is known as the Balaban 11-cage (named after Alexandru T. Balaban who found it in 1973; Hartsfield and Ringel seem to refer to a 12-cage on 126 vertices, but this seems to be a typographical error). These are the only known g-cages (according to Hartsfield and Ringel, page 79).

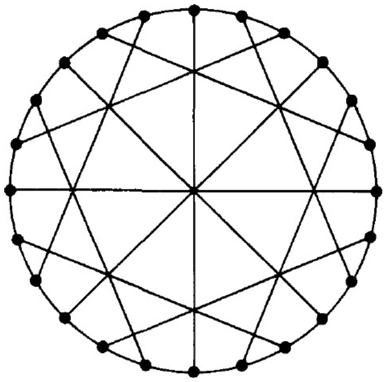


Figure 4.2.6. The McGee Cage

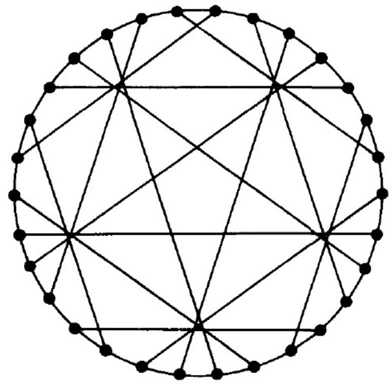


Figure 4.2.7. The Tutte-Coxeter Cage

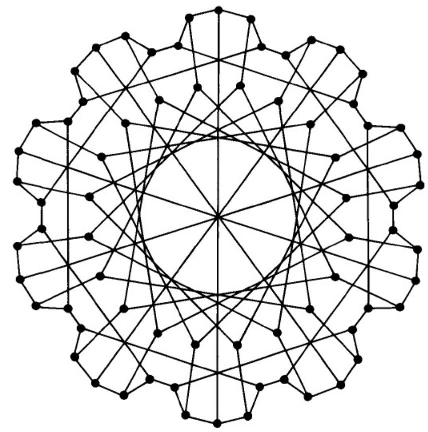


Figure 4.2.8. Balaban's 10-Cage

Note. As a passing bit of trivia, the Institute for Combinatorics and Its Applications has the Tutte-Coxeter graph (i.e., the unique 8-cage) on the cover of its *Bulletin of the Institute for Combinatorics and Its Applications*:

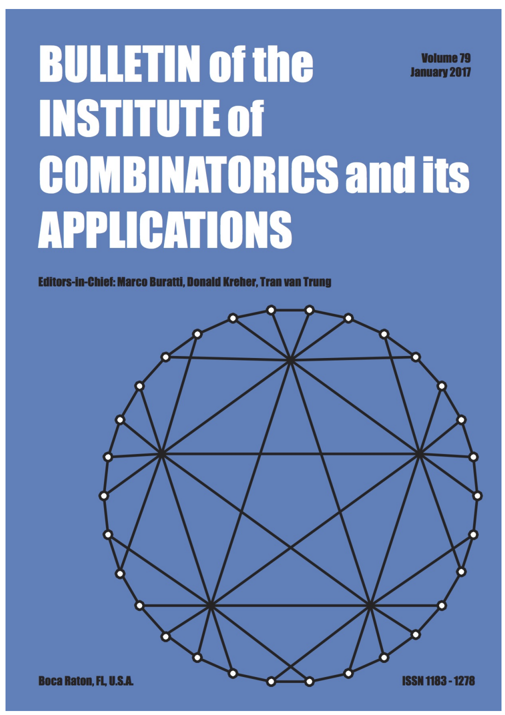


Image from the *Bulletin of the Institute for Combinatorics and Its Applications* webpage (accessed 5/1/2022)