Section 4.3. Ramsey Theory

Note. In this section we consider edge-colorings of complete graphs with two colors. We look for conditions (in terms of the number of vertices of the complete graph) such that the two subgraphs induced by monochromatic edges must contain complete subgraphs of a given size. We start with a small example (stated as a lemma).

Lemma 4.3.A. If the edges of K_6 are colored with two colors, then there must be a monochromatic triangle. Also, K_6 is minimal complete graph with respect to this property.

Note. In popular-level applications of graph theory, Lemma 4.3.A is sometimes called the "Theorem on Friends and Strangers." The setting is a party with six people. Two of the party attendees are *mutual strangers* if they have never met before and are *mutual acquaintances* if they have met before. The Theorem on Friends and Strangers states that: "In any party of six people either at least three of them are (pairwise) mutual strangers or at least three of them are (pairwise) mutual acquaintances." The six attendees are represented by vertices of K_6. Edges joining mutual strangers are colored red and edges joining mutual acquaintances are colored blue. By Lemma 4.3.A, the K_6 must contain a red or a blue triangle. That is, there are at least three mutual strangers (corresponding to a red triangle) or at least three mutual acquaintances (corresponding to a blue triangle), as claimed by the Theorem on Friends and Strangers. This information is from the Wikipedia webpage on the Theorem of Friends and Strangers (accessed 5/7/2022).

Note. In the spirit of Lemma 4.3.A, it is also known that if the edges of K_{18} are colored with two colors, then there must be a monochromatic K_4 and K_{18} is the minimal complete graph with respect to this property, as we will discuss below. These are examples of a more general result. It is due to Frank P. Ramsey (February 22, 1903–January 19, 1930).

Image from the MacTutor History of Mathematics Archive biography of Ramsey (accessed 5/7/2022)

Ramsey worked in philosophy, mathematics, and economics. He was a friend of Ludwig Wittgenstein while at the Trinity College of Cambridge University (London). In his paper "On a Problem of Formal Logic," *Proceedings of the London Mathematical Society* **30**(1), 264–286 (1930) (a version is available online on William Gasarch's University of Maryland webpage; the 1928 date on the paper reflects the fact that it was read to the society on December 13, 1928), Ramsey presents the main result of this section. However, as the title of the paper suggests, this was not the main focus of the paper, though this is probably his most famous result, and the theorem appears more in passing than as the focus of his work which was addressing a decidability problem in first order logic. His related work on a theory of types (related to the work of Russell and Whitehead in their three volume *Principia Mathematica* of the early 1900s; this work appeared as F. P. Ramsey, F.P. "The Foundations of Mathematics," *Proceedings of the London Mathematical Society*, **25**(1), 338-384 (1926)) was later used by Kurt Gödel in his work on incompleteness. For details on this work in mathematical foundations, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.6. Completeness and Categoricalness and my online presentation for Great Ideas in Science 1 and 2 (BIOL 3018 and BIOL 3028) on Introduction to Math Philosophy and Meaning. Ramsey died of complications following abdominal surgery for liver problems in 1930. He was a month shy of his 27th birthday. This biographical information is from the Wikipedia page on Frank Ramsey (accessed 5/7/2022). Ramsey's Theorem relates to two colorings and monochromatic complete subgraphs.

Theorem 4.3.1. Ramsey's Theorem.
For every number n, there is a number $r(n)$ such that any edge-coloring of the complete graph with $r(n)$ vertices using red and blue must contain either a red K_n or a blue K_n.

Note. Notice that Ramsey's Theorem (Theorem 4.3.1) does not actually give the *value* of $r(n)$, but merely guarantees the existence of of such a value. We will get Ramsey's Theorem out as a corollary to a more general result which we will prove (Theorem 4.3.2 below). To motivate the generalization, we consider another lemma.

Lemma 4.3.B. If the edges of K_9 are colored with red and blue, then there is a subgraph of this K_9 that is either a red K_3 or a blue K_4. Also, K_9 is minimal complete graph with respect to this property.

Definition. The *Ramsey number* $r(m, n)$ is the smallest number with the property that any edge-coloring of the complete graph with $r(m, n)$ vertices using red and blue must contain a red K_m or a blue K_n.

Note 4.3.A. Lemma 4.3.B shows that $r(3, 4) = 9$. When $m = 1$ or $m = 2$, we can easily find $r(m, n)$. We have $r(1, n) = 1$ since any edge-coloring of K_1 with two colors contains a red K_1 or a blue K_n, because $K_m = K_1$ has no edges so that all edges of $K_m = K_1$ are red (vacuously). We also have $r(2, n) = n$ because any edge-coloring of K_n with red and blue contains either a red K_2 (i.e., a red edge) or a blue K_n (when no edges are red and all edges are blue). Other than these two easy classes of Ramsey numbers, very few other Ramsey numbers are precisely known (though bounds exist which give a range of possible values for other Ramsey numbers). Hartsfield and Ringel (copyright 1990) give the following as the only known precise values (see their page 84):

$$r(1, n) = 1 \text{ and } r(2, n) = n \text{ as just explained,}$$

$$r(3, 3) = 6 \text{ by Lemma 4.3.A, } r(3, 4) = 9 \text{ by Lemma 4.3.B,}$$

$$r(3, 5) = 14,\ r(3, 6) = 18,\ r(3, 7) = 23,\ r(3, 9) = 36, \text{ and } r(4, 4) = 18.$$

The values of $r(3, 5)$ and $r(4, 4)$ are established in Graph Theory 2 (MATH 5450) in Section 12.3. Ramsey's Theorem. A document giving the latest results on these studies is “Small Ramsey Numbers” document by Stanisław P. Radziszowski (Revision #16: January 15, 2021; accessed 5/7/2022). It includes two more known Ramsey numbers. The fact that $r(3,8) = 28$ was shown in B.D. McKay and Zhang Ke Min's “The Value of the Ramsey Number $R(3,8)$,” *Journal of Graph Theory*, **16**, 99–105 (1992). The fact that $r(4,5) = 25$ was shown in B.D. McKay, and S. Radziszowski's “$R(4,5) = 25$,” *Journal of Graph Theory* **19**(3), 309-322 (1995). The Wikipedia webpage on Ramsey's theorem includes some of the same information; it is less academic, but might be updated faster in the event of new discoveries (accessed 5/7/2022). We now state and prove an inequality related to Ramsey numbers.

Theorem 4.3.2. For every m and n, there exists the Ramsey number $r(m,n)$ such that edge-coloring $K_{r(m,n)}$ with red and blue implies that $K_{r(m,n)}$ contains either a red K_m or a blue K_n. Furthermore, $r(m,n)$ satisfies the inequality $r(m,n) \leq r(m-1,n) + r(m,n-1)$.

Note. With $m = n$ in Theorem 4.3.2 (so that $r(n,n) = r(n)$), we have that Ramsey's Theorem (Theorem 4.3.1) follows from Theorem 4.3.2. We now consider a result related to those above, except that it involves complete bipartite graphs instead of complete graphs.

Lemma 4.3.C. If the edges of $K_{5,5}$ are colored with two colors, there will be a monochromatic $K_{2,2}$.