Chapter 5. Counting

Section 5.1. Counting 1-Factor

Note. In this section we count the number of different subgraphs of a particular type in some graph (usually a type of complete bipartite or complete graph). Throughout, we use the Fundamental Counting Principle; see my online notes for Applied Combinatorics and Problem Solving (MATH 3340) on Section 1.1. The Fundamental Counting Principle. Hartsfield and Ringel state the results of this section as "Problems," but they are presented here as theorems.

Theorem 5.1.A. There are $n!$ 1-factors in $K_{n,n}$.

Theorem 5.1.B. There are $\dfrac{n!}{2(n-k-1)!}$ different subgraphs of K_n isomorphic to path P_k.

Note. Recall that $\binom{n}{k}$ is the number of *combinations* of k objects taken from a set of size n. Algebraically, it is $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. For more details, see my online notes for Foundations of Probability and Statistics-Calculus Based (MATH 2050) on Section 2.2. Counting Methods; see Note 2.2.D.

Theorem 5.1.C. There are $n\binom{n-1}{3}$ different subgraphs of K_n isomorphic to $K_{1,3}$.

Theorem 5.1.D. There are $\dfrac{(2h-1)!}{2^{h-1}(h-1)!}$ different 1-factors in K_{2h}.

Note 5.1.A. Next, we consider the number of 1-factors in the graph "$K_{n,n}$ minus a 1-factor." This might seem related to Theorem 5.1.A where we saw that there are $n!$ 1-factors in $K_{n,n}$ itself, but as we will see by formula given in Theorem 5.1.E, the number of 1-factors in $K_{n,n}$ minus a 1-factor is not a simple factorial. Interest in this question can be motivated by the Hatcheck Problem. Suppose a group of n people go to a restaurant and check their hats. How many ways can their hats be returned to them so that no person gets the correct hat back? The Letter Problem is similar. In this, a secretary has typed n letters and addressed n envelopes for the letters. The letters are matched with their corresponding envelope in a pile on the desk. The janitor knocks the letters and envelopes off the desk. How many ways can the janitor pick up the letters and envelopes so that every letter is associated with the wrong envelope? Both of these problems are examples of a *derangement* of n objects. Both involve matching up an object in one category with an object in another category (the objects correspond to vertices, the categories correspond to the partite sets in $K_{n,n}$, and the matching corresponds to the edges of a 1-factor). So the number of 1-factors in the graph $K_{n,n}$ minus a 1-factor is the number of derangements of n objects.

Theorem 5.1.E. There number of different 1-factors in $K_{n,n}$ minus a 1-factor is

$$n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-1}}{(n-1)!}+\frac{(-1)^n}{n!}\right).$$

Note 5.1.B. We know from Calculus 2 that the series representation for e^x is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

See my online Calculus 2 notes on Section 10.8. Taylor and Maclaurin Series. So

$$\begin{aligned} e^{(-1)} &= 1 + \frac{(-1)}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \cdots + \frac{(-1)^n}{n!} + \cdots \\ &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!} + \cdots . \end{aligned}$$

Therefore, for n large the number of 1-factors in $K_{n,n}$ minus a 1-factor satisfies $D_n \approx n!/e$.

Note. Hartsfield and Ringel antoher example of a derangement concerning a test consisting of 10 questions that are to be matched with 10 answers. Every collection of answers corresponds to a 1-factor in $K_{10,10}$. One of these 1-factors gets all questions correct. If we remove this 1-factor from $K_{n,n}$ then we remove all edges corresponding to correct answers and all of the edges left correspond to incorrect answers. Hence, a 1-factor in $K_{n,n}$ minus a 1-factor is the number of ways to complete the test and get all of the questions wrong. By Theorem 5.1.A, the number of ways to complete the test is 10!. If a student answers the questions randomly, then the probability of getting all questions correct is 1/10! and the probability of all questions wrong is

$$\frac{D_n}{10!} \approx \frac{10!/e}{10!} = \frac{1}{e}.$$

Since $1/e \approx 1/3$, guessing the matches on such a test gives a probability of roughly 1/3 of getting *all* questions wrong!