Chapter 6. Labeling Graphs

Section 6.1. Magic Graphs and Graceful Trees

Note. In this section, we define magic graphs and argue that all complete symmetric bipartite graphs $K_{n,n}$ is magic except for $n = 2$. We relate decompositions into Hamilton cycles to magic graphs. We define antimagic graphs and give two conjectures concerning them. We define graceful and consecutive trees and relate them to other decompositions and to each other.

Definition. A graph G with q edges is *magic* if the edges of G can be labeled by the numbers $1, 2, 3, \ldots, q$ so that the sum of the labels of all the edges incident with any vertex is the same.

Note. Two examples of magic graphs are given in Figure 6.1.2. The sum of the labels of the edges incident a vertex is 21 for the graph on the left and 24 for the graph on the right.

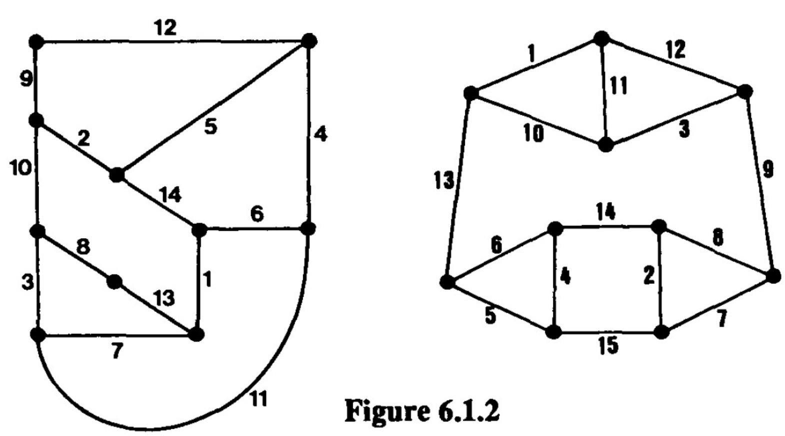


Figure 6.1.2

Note. A "magic square" is an $n \times n$ array containing each of $1, 2, \ldots, n^2$ exactly once, such that the sums of edges and the sums of the columns are all the same. It is well known that an $n \times n$ magic square exists for all $n \geq 3$. We can use an $n \times n$ magic square to show that the complete bipartite graphs $K_{n,n}$ are magic for $n \geq 3$. For example, the 3×3 magic square in Figure 6.1.4 and the 4×4 magic square in Figure 6.1.5 yield the labelings of $K_{3,3}$ in Figure 6.1.3 and of $K_{4,4}$ in Figure 6.1.6. We can see from Figure 6.1.3 that the edges incident to the vertices in one partite set are labeled with the entries of the rows of the magic square, and the edges incident to the vertices in the other partite set are labeled with the vertices of the columns of the magic square. We can generalize this by enumerating the vertices in the partite sets and labeling the edge from the ith vertex of the first partite set to the jth vertex of the second partite set with the entry in the ith row and j column of the magic square. This observation results in the next theorem.

2	7	6
9	5	1
4	3	8

Figure 6.1.4

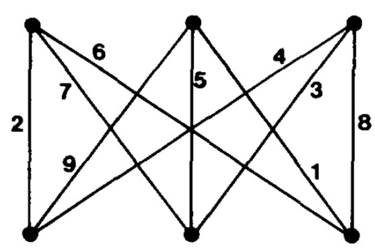


Figure 6.1.3

3	11	14	6
8	5	9	12
10	2	7	15
13	16	4	1

Figure 6.1.5

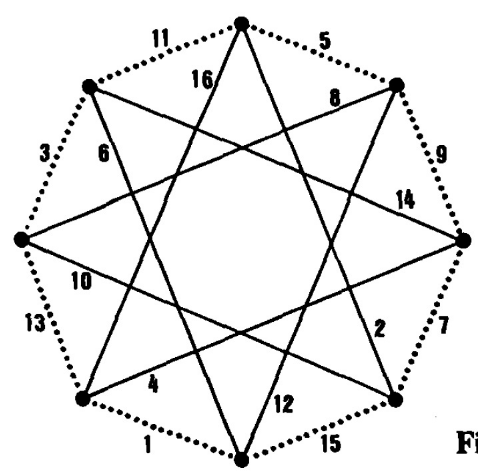


Figure 6.1.6

Theorem 6.1.1. $K_{n,n}$ is magic for all $n \neq 2$.

Note. Notice in Figure 6.1.6 that $K_{4,4}$ can be decomposed into two Hamilton cycles; one of the cycles is given by solid edges and the other by dotted edges. In fact, a decomposition of a bipartite graph into Hamilton cycles implies that the graph is magic, as we now prove.

Theorem 6.1.2. If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.

Note. The labeling of the bipartite graph $K_{5,5}$ is illustrated in Figure 6.1.7. Notice the edges with odd-number labels are on the outer Hamilton cycle, and the edges with even-number labels are on the inner Hamilton cycle.

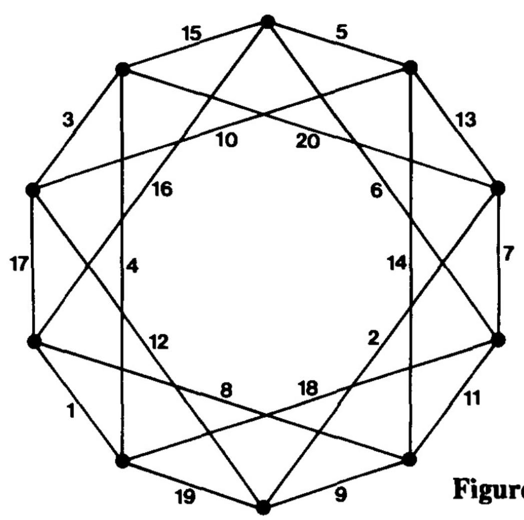


Figure 6.1.7

Note. Bipartite graphs are not the only graphs that are magic. Figure 6.1.9 gives a non-bipartite graph (notice the existence of 3-cycles) which is magic. The sum of the edge labels incident to each vertex is 34. There is a typo in the figure in the book; one edge is labeled 18 and it should be labeled 16 (as it is here).

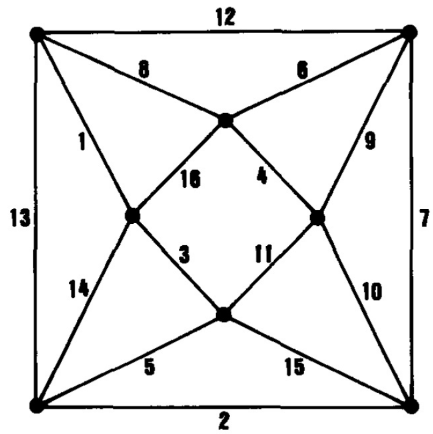


Figure 6.1.9

Note. Complete graphs have been classified in terms of the property of being magic. In Exercise 6.1.6 it is to be shown that if $n \equiv 0 \pmod 4$ then K_n is not magic. In fact, for all $n \geq 6$ with $n \not\equiv 0 \pmod 4$, K_n is a magic graph. This result appears in B. M. Stewart's "Supermagic Complete Graphs," *Canadian Journal of Mathematics*, **19**, 427–438 (1967); the term "supermagic" used in the title means the same as our term "magic." This paper is online on the Cambridge University Press website (accessed 1/4/2023). The next theorem allows us to potentially construct magic graphs from magic spanning subgraphs.

Theorem 6.1.3. If a graph G is decomposable into two magic spanning subgraphs G_1 and G_2 where G_2 is regular, then G is magic.

Definition. A graph G with q edges is *antimagic* if the edges of G can be labeled by the numbers $1, 2, 3, \ldots, q$ so that the sum of the labels of all the edges incident with any vertex is different from the sum of the labels at any other vertex.

Note. Figure 6.1.11 gives four examples of antimagic graphs.

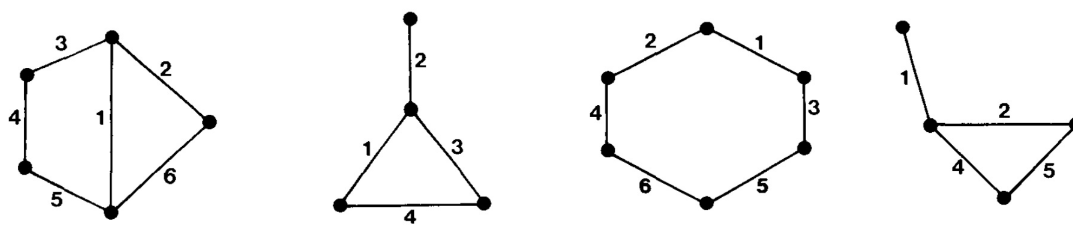


Figure 6.1.11

Note. Not nearly as much is known about antimagic graphs as is known about magic graphs. Two conjectures concerning antimagic graphs are:

Conjecture 6.1.A. Every tree different from K_2 is antimagic.

Conjecture 6.1.B. Every connected graph different from K_2 is antimagic.

Since every tree is (by definition) a connected graph, the second conjecture is much stronger than the first.

Definition. Consider a tree with n vertices (and so $n-1$ edges by Theorem 1.3.2). If it is possible to label the vertices by $1, 2, 3, \ldots, n$ and the edges by $1, 2, 3, \ldots, n-1$ so that the label on any edge equals the difference between the labels of the two endpoints, then the tree is *graceful*.

Note. The idea of a graceful labeling of a simple graph in general (not just a tree) can similarly be defined. The study of graceful graphs and graceful labelings is an area more widely studied than antimagic graphs (and more studied than magic graphs, in the opinion of your humble instructor). Evidence of this is the fact that there is a Wikipedia page for graceful labelings and there is not one for antimagic graphs (as of 1/4/2023).

Note. Figures 6.1.13 and 6.1.14 give graceful labelings of two small trees.

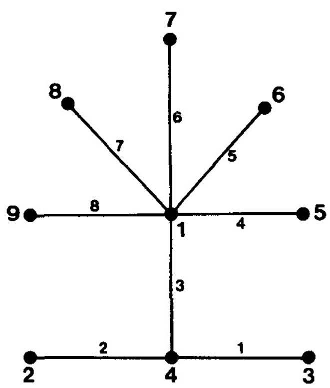


Figure 6.1.13

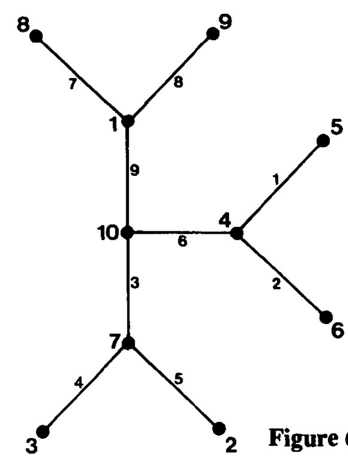
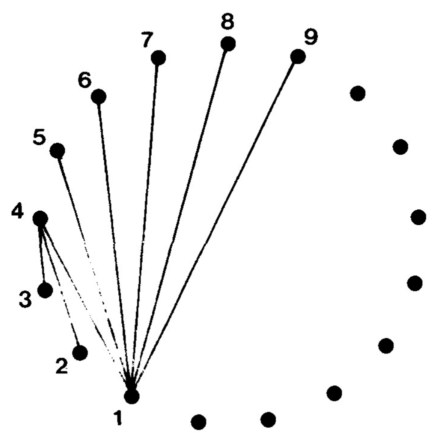


Figure 6.1.14

The Graceful Tree Conjecture claims that every tree is graceful. This conjecture is due to Gerhard Ringel (one of the authors of this book), though the Wikipedia page mentioned above credits the conjecture to both Ringel and Anton Kotzig.

Note. Graceful labelings and the "turning trick" are related to decomposition problems. We mentioned difference methods in passing in Section 2.3. Decompositions and Hamilton Cycles; more details are given in my online notes for Design Theory (not an official ETSU class) on Section 1.7. Cyclic Steiner Triple Systems and in my online notes for graduate level Graph Theory 1 (MATH 5340) on Supplement. Graph Decompositions: Triple Systems. Below is an image of a labeling of the nine vertices of a tree. Ignoring the other vertices, this is a graceful labeling, since the (absolute value of) the associated "differences" on the eight edges are 1, 2, ..., 8. Now there are a total of 17 vertices here. If we label all the vertices as 1, 2, ..., 17 then total possible differences associated with any edge is 1, 2, ..., 8. So if we apply the "turning trick" to the pictured tree and rotate it around then it will produce all edges of K_{17}. That is, this graceful labeling of the given tree implies that K_{17} can be decomposed into copies of the tree (17 copies, in fact).

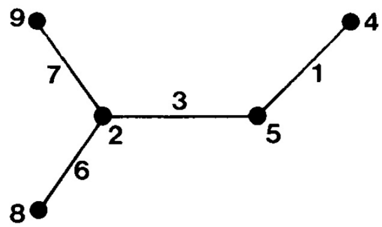


This approach works for any graceful tree, so that we have the following.

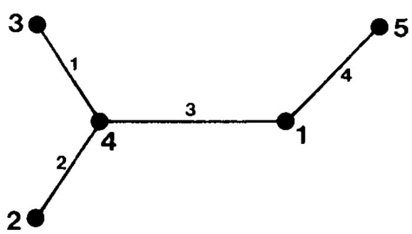
Theorem 6.1.4. If a tree T with n edges is graceful, then the complete graph K_{2n+1} is decomposable into $2n+1$ trees, each isomorphic to the given tree T.

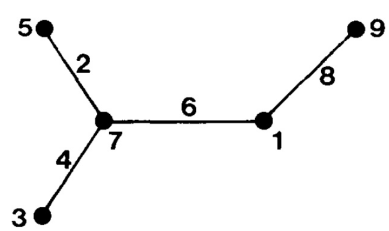
Definition. Consider a labeling of the set of vertices and edges of tree T with n edges by the consecutive integers $1, 2, 3, \ldots, 2n-1$. The labeling of T is a *consecutive* labeling of T if the label of every edge equals the absolute value of the difference of the labels of the two vertices of the edge.

Note. Here is an example of a consecutive labeling of a small tree.



In fact, if a tree with n edges has a graceful labeling, then it can be used to generate a consecutive labeling of the tree. Below (left) is a graceful labeling of the tree above. We modify this graceful labeling by replacing vertex label k with the label $2k-1$, so that all vertex labels are then odd numbers. Next, we label each edge by the (absolute value of) the difference of the endpoints, and then each label will be even and will be twice the original label. This last claim follows by considering vertices originally labeled j and k, so that the edge joining them in the graceful labeling is $|j-k|$. Under the new labeling, the vertices become labeled $2j-1$ and $2k-1$, respectively, and the new edge label is $|(2j-1)-(2k-1)| = 2|j-k|$. The consecutive labeling of the tree on the left produced by this process is given on the right. This results, in general, in a consecutive labeling and our last theorem of this section.





Theorem 6.1.5. If a tree T is graceful, then T is consecutive.

Note. It is conjectured that all trees have consecutive labelings. Of course this is weaker than the conjecture that all trees are graceful (Conjecture 6.1.A), by Theorem 6.1.5. Again, consecutive labelings, like graceful labelings, can defined for general simple graphs, not just for trees.

Revised: 1/4/2023