# Section 6.2. Conservative Graphs

**Note.** In this section, we introduce directed graphs and use them to consider "flows" through a graph by introducing labels on the arcs of the directed graphs. We define conservative and strongly conservative graphs. We give some properties that insure that a graph is conservative or strongly conservative. We prove that all (sufficiently large) complete graphs and wheels are conservative graphs.

**Definition.** A *directed graph* (or *digraph*) $D$ is a pair of sets $(V, A)$ where $V$ is nonempty, and $A$ is a (possibly empty) set of ordered pairs of elements of $V$. The elements of set $A$ are *arcs* of $D$. For arc $(u, v)$, vertex $u$ is the *tail* and $v$ is the *head*.

**Note.** When drawing a digraph, we represent the vertices as points as usual, and represent the arcs by curves with little arrow heads which indicate a direction from the tail to the head of the arc.

**Definition.** A graph with $q$ edges which can be directed (that is, the edges are given an *orientation* to make them arcs) and the arcs labeled with with numbers $1, 2, 3, \ldots, q$ such that, for each vertex, the sum of the labels on the arcs coming into a vertex is the same as the sum of the labels on the arcs going out of that vertex is a *conservative* graph.

**Note.** Figure 6.2.1 is a conservative graph, as shown by the orientation and labeling given in Figure 6.2.2.

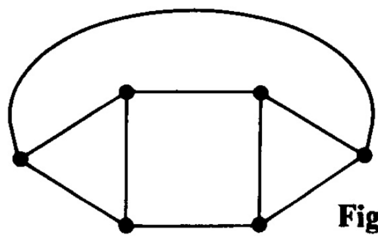

Figure 6.2.1

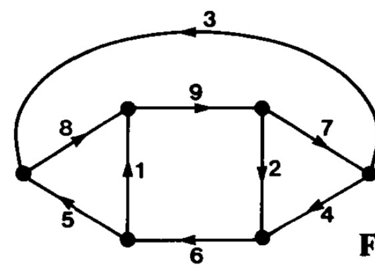


Figure 6.2.2

Notice that we could use the idea of a conservative graph to model the flow of some fluid through a network of pipelines (though the constraint of forcing the labels to be $1, 2, \ldots, q$ is artificial in this case). The property of conservation at each vertex, in the setting of electrical flow through a network of wires, is often called Kirchhoff's Current Law:

**Kirchhoff's Current Law.** In a directed labeling of a conservative graph, the sum of the incoming labels is equal to the sum of the outgoing labels at each vertex.

We will refer to the sum of the incoming labels minus the sum of the outgoing labels as the *directed sum* at a vertex. Then Kirchhof's Current Law can be stated as: In a directed labeling of a conservative graph, the directed sum is zero at each vertex.

**Note.** The dodecahedron graph is conservative, as shown in Figure 6.2.3. The labeling was found by Sharon Cabaniss using a computer search.

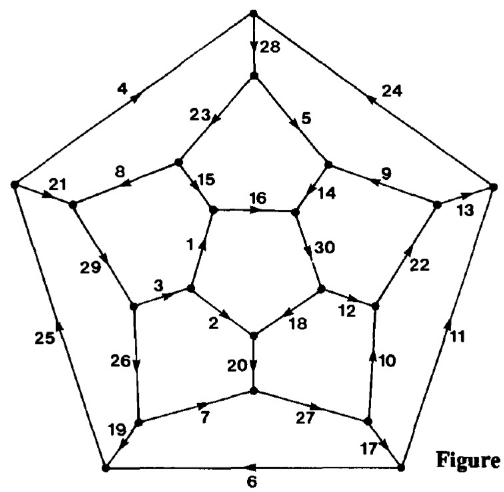


**Figure 6.2.3. The Cabaniss dodecahedron**

The next theorem gives a condition that can be used to show some graphs to be conservative.

**Note.** We'll see similarities between the results of Section 6.1 on magic graphs and the results in this section on conservative graphs. The following is similar to Theorem 6.1.2 (though it only applied to bipartite graphs).

**Theorem 6.2.1.** If graph $G$ is decomposable into two Hamilton cycles, then $G$ is conservative.

**Note.** In the proof of Theorem 6.2.1, we chose a vertex $a$ as a starting vertex and then oriented and labeled the two Hamilton cycles. After showing that the Kirchhoff Current Law holds at the vertices other than $a$, we then verified that it also holds at $a$ itself. It turns out that this last step is unnecessary, as we prove next.

**Theorem 6.2.2. Kirchhoff's Global Current Law.**
If $G$ is a labeled, directed graph such that Kirchhoff's Current Law at every vertex of $G$ except a particular vertex $a$, then Kirchhoff's Current Law also holds at the vertex $a$.

**Definition.** A graph $G$ with $q$ edges is *strongly conservative* if for every number $h$ there exists a directed labeling of the edges of $G$ with the numbers $h+1, h+2, \ldots, h+q$ such that Kirchhoff's Current Law holds at every vertex.

**Note.** The graph if Figure 6.2.4 below is conservative. Since each vertex is in-degree two and out-degree two, then is is also strongly conservative.

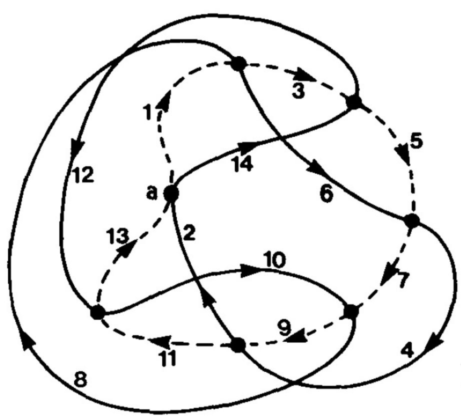

**Figure 6.2.4**

The next result is similar to Theorem 6.1.3 for magic graphs.

**Theorem 6.2.3.** If $G$ is decomposable into two subgraphs $H_1$ and $H_2$, and if $H_1$ is conservative, and $H_2$ is strongly conservative, then $G$ is conservative. Moreover, if both $H_1$ and $H_2$ are strongly conservative, then $G$ is strongly conservative.

**Note.** In Theorem 6.2.1, we considered a graph $G$ decomposable into two Hamilton cycles. Notice that if we add $h$ to all arc labels in the conservative labeling, the sum of the labels of arcs going into a given vertex of $G$ increases by $2h$ and the sum of the labels of arcs leaving that vertex also increase by $2h$, so that this new labeling also has directed sum of zero. So we can conclude that $G$ is not just conservative, but is strongly conservative. We state this as a new result:

**Theorem 6.2.1*.** If $G$ is decomposable into two Hamilton cycles, then $G$ is strongly conservative.

**Note.** We now give a condition that guarantee that a graph is strongly conservative.

**Theorem 6.2.4.** If $G$ is a graph with $n$ vertices, where $n$ is odd, and $G$ is decomposable into three Hamilton cycles, then $G$ is strongly conservative.

**Note.** The labeling of the arcs used in the proof of Theorem 6.2.4 is illustrated in Figure 6.2.6.

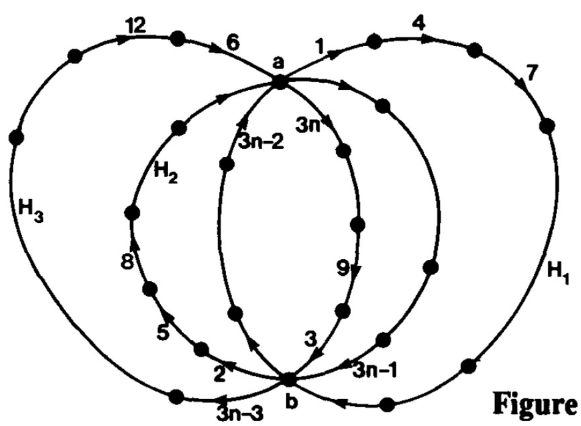


**Figure 6.2.6**

**Note.** We conclude this section by proving that (sufficiently large) complete graphs and wheels are conservative graphs.

**Theorem 6.2.5.** If $n$ is odd, $n \geq 5$, then $K_n$ is conservative.

**Theorem 6.2.6.** For $n \geq 3$, the wheel with $n$ spokes, $W_n$, is conservative.

**Theorem 6.2.7.** If $n$ is even, $n \geq 4$, then $K_n$ is conservative.

**Note.** Combining Theorems 6.2.5 and 6.2.7, we see that all complete graphs $K_n$, where $n \geq 4$, are conservative graphs.

*Revised: 1/15/2023*