Section 8.2. The Four Color Theorem

Note. In this section we give a brief description of the Four Color Theorem. We give definitions and some results related to four colorings of maps. In particular, we state the Four Color Theorem in terms of duals of graphs. However, we do not address any components of the proof of the Four Color Theorem.

Definition. In a plane drawing of a multigraph, a region determined by only two edges is a *lune*. A plane drawing of a connected, bridgeless, planar multigraph is a *map*. A map in which each vertex is degree 3 (i.e., a "cubic" map) is a *normal map*.

Note. Figure 8.2.1 below shows a normal map. In Figure 8.2.2, a plane drawing of a graph is given that is not a map because it contains a bridge. Notice that a bridge in a map implies that there is a border between a region and itself (thus our interest in bridgeless graphs).

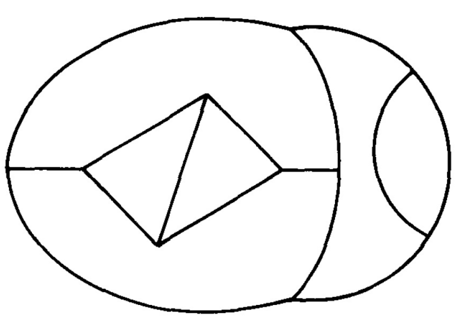

Figure 8.2.1

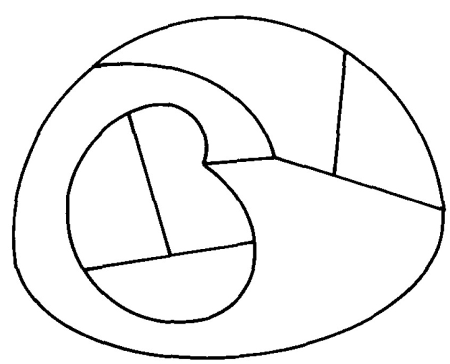

Figure 8.2.2

Definition. The regions of a map are called *countries*. Two countries are *adjacent* if they have an edge in common. Sometimes the edges are called *borders*. A *coloring* of a map M is an assignment of a color to each of the countries of M so that no two adjacent countries have the same color.

Note. Recall that two edges of a graph (or multigraph) are "adjacent" if they are incident with the same vertex. A proper edge coloring of a graph (or multigraph) G is an assignment of a color to each edge of G so that no two adjacent edges receive the same color. The next result, due to Peter Tait (April 28, 1831–July 4, 1901) in 1880, relates four colorings of countries of a normal map to a proper three coloring of the edges of the graph.

Theorem 8.2.1. If a normal map has a coloring of the countries with four colors, then the edges of the map can be properly colored by three colors.

Note. In fact, Tait also proved the converse of Theorem 8.2.1. We first present a lemma, and then will use it to prove the converse of Theorem 8.2.1.

Lemma 8.2.2. Given a finite number of simple closed curves (cycles) in the plane that only intersect at points (not along segments of the curves), the regions defined by these curves are colorable by two colors.

Theorem 8.2.3. If the edges of a normal map can be properly colored by three colors, then the countries of the map can be colored by four colors.

Note 8.2.A. Hartsfield and Ringel declare the following two theorems an "interesting labeling" of the vertices by two colors (see pare 159). They give a condition equivalent to the proper three coloring of the edges of a normal map and therefore, by Theorems 8.2.1 and 8.2.3, a condition equivalent to a coloring of the countries of the normal map with four colors.

Theorem 8.2.4. If in a normal map the edges are properly colored by three colors, then the vertices can be labeled black and white so that around any given country, the number of black vertices minus the number of white vertices is always a multiple of three.

Note. The converse of Theorem 8.2.4 holds, as follows. We give a proof that relies heavily on a picture.

Theorem 8.2.5. If in a normal map M, the vertices can be labeled by black and white so that around each country in M the number of black vertices minus the number of white vertices is a multiple of three, then the edges of M can be properly colored by three colors.

Note. The Four Color Theorem is, possibly, the best known graph theory problem. It is simple to state, has a rich history including an infamous wrong proof, and a solution that relied heavily on a computer search that inspired much debate in the mathematical community. I have a lengthy history in my online notes for graduate level Graph Theory 2 (MATH 5450) on Supplement. The Four-Color Theorem: A History, Part 1 and Supplement. The Four-Color Theorem: A History, Part 2. Notes addressing the mathematics of the Four Color Problem are in my online notes for Graph Theory 2 on Section 11.1. Colourings of Planar Maps and Section 15.2. The Four-Colour Theorem.

Theorem 8.2.6. Four Color Theorem.
In every normal map the countries are colorable by four colors.

Note. By Theorems 8.2.1 and 8.2.3, the following is equivalent to the Four Color Theorem.

Theorem 8.2.7. Four Color Theorem.
In every planar cubic bridgeless graph, the edges are properly colorable be three colors.

Note. As opposed to considering normal maps themselves, starting in the early 20th century, the Four Color Problem was addressed by considering vertex colorings of the "dual" of the graph. The dual is a multigraph (possibly with multiple edges, but no loops)

Definition. Given a normal map M, we define the multigraph $G(M)$ with the following properties. The vertices of $G(M)$ are the countries of M. Two vertices of $G(M)$ are adjacent if the countries are adjacent in M. A planar drawing of $G(M)$ is the *dual* of normal map M.

Note 8.2.B. It is intuitively clear that if M is a normal map then $G(M)$ is a planar graph. This is established a bit more formally in graduate level Graph Theory 2 (MATH 5450) in Section 10.2. Duality; see Lemma 10.2.A. So we are justified in the previous definition of referring to "a planar drawing of $G(M)$." We can also define the dual of a general planar graph. Figure 8.2.12 illustrates a general planar graph (with white vertices) and its dual (with black vertices). Notice that neither graph is a normal map since neither is a cubic graph. In fact, the dual of a normal map is a *triangulation*; that is, a planar graph where each region contains three edges. If D is a plane drawing of a maximal planar graph with at least four vertices, then the dual $G(D)$ is a normal map (we leave this claim as Exercise 8.2.A; hints are given on page 163). We can the state the Four Color Theorem in purely graph theoretical terms.

Theorem 8.2.8. Four Color Theorem.
The vertices of every maximal planar graph with at least four vertices can be colored with at most four colors.

Note. Since every planar graph is a subgraph of a maximal planar graph, we can state the Four Color Theorem in terms of general planar graphs, as follows.

Theorem 8.2.9. Four Color Theorem.
The vertices of every planar graph can be colored by at most four colors.

Note. In the terminology of Section 2.1. Vertex Colorings, we could reword the Theorem 8.2.9 version of the Four Color Theorem as: The chromatic number of every planar graph is at most four.

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