

## Section 8.3. The Five Color Theorem

**Note.** In this section we give a (relatively lengthy) proof that every normal map can be colored by five colors. Since there is no concise readable proof of the Four Color Theorem, so we present a proof of this weaker result. We need a preliminary lemma.

**Lemma 8.3.1.** No map in the plane has five mutually adjacent countries.

**Note.** We now prove the Five Color Theorem. The proof technique is the same as that used in the 1976 proof of the Four Color Theorem by Appel and Haken (see my supplement for Graph Theory 2 [MATH 5450] on [Supplement. The Four-Color Theorem: A History, Part 2](#) for some informal details). The approach is to give a proof by contradiction. Under the assumption that the Five Color Theorem is false, there exists some normal map that is not five colorable. Among all such non five colorable normal maps, there is a minimal one (that is, one with the fewest number of countries). We concentrate on such a map and we can use the minimality in the proof. Notice that the minimality implies that if we reduce the number of countries (by eliminating a border, for example) then the new map can be colored using five colors. We will then modify this five coloring of the new map to produce a five coloring of the original map.

**Theorem 8.3.2. The Five Color Theorem.**

The countries of every normal map in the plane can be colored by five colors.

**Note.** A proof of the Five Color Theorem is also given in graduate level Graph Theory 2 (MATH 5450). See my online notes for this class on [Section 11.2. The Five-Colour Theorem](#). The proof given there is shorter than the one given here, but it assumes a larger body of knowledge.

*Revised: 12/30/2022*