Chapter 9. Measurements of Closeness to Planarity

Section 9.1. Crossing Number

Note. In this section we state (without proof) Kuratowksi's Theorem, which classifies planar graphs. We define crossing and simple drawing, and use these ideas to define the crossing number of a graph. Examples are given of the calculation of crossing numbers for certain graphs, and a bound on the crossing number of a complete bipartite graph is shown (in Theorem 9.1.5).

Definition. A *subdivision* of a graph results when some vertices of degree two are added to edges of the graph (that is, some of the edges are subdivided into paths of length greater than one). A graph is considered a subdivision of itself.

Note. Figure 9.1.1 is a subdivision of K_5 and Figure 9.1.2 is a subdivision of $K_{3,3}$.

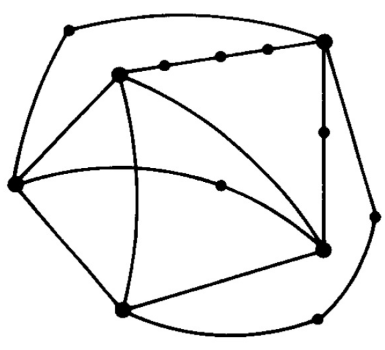

Figure 9.1.1.

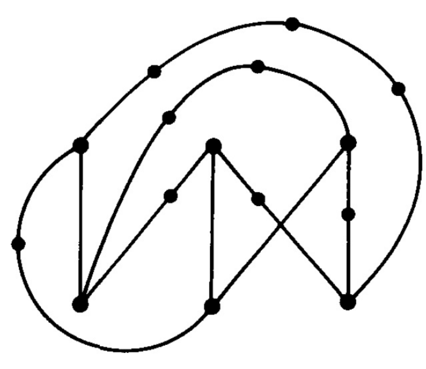

Figure 9.1.2.

Note 9.1.A. We observe (without proof) that:

1. If a graph is not planar then a subdivision of the graph is not planar.

2. If a graph contains a nonplanar subgraph, then the graph is not planar.

These two "observations," along with Theorems 8.1.4 and 8.1.6 (which tell us that K_5 and $K_{3,3}$ are not planar), imply the following.

Theorem 9.1.1. If a graph G contains a subgraph that is a subdivision of K_5 or of $K_{3,3}$ then G is not planar.

Note. In fact, the converse of Theorem 9.1.1 holds. In 1930, Kazimierz (aka. "Casimir") Kuratowski gave the classification of nonplanar graphs in terms of subgraphs related to K_5 and $K_{3,3}$ in "Sur le poblème des courber gauches en topologie," *Fundamenta Mathematicae*, **15**, 271–283. A copy is available online in French (accessed 12/31/2022).

Kazimierz Kuratowski (February 2, 1896 - 18 June 18, 1980)

Photo from MacTutor History of Mathematics Archive (accessed 12/31/2022). We explore Kuratowski's result in graduate level Graph Theory 2 (MATH 5450) in Section 10.5. Kuratowski's Theorem where a proof is given of Wagner's Theorem, which is equivalent to Kuratowski's Theorem (see Note 10.5.C in those notes). In this class, we simply state Kuratowski's Theorem.

Theorem 9.1.2. Kuratowski's Theorem.
If G is a nonplanar graph, then G contains a subgraph is a subdivision of K_5 or of $K_{3,3}$.

Note. The next result is equivalent to Kuratowski's Theorem (a claim we do not prove).

Corollary 9.1.A. Assume that G is a graph with the property that G is not planar, and for every edge e of G, the graph $G - e$ is planar. Then G is either a subdivision of K_5 or a subdivision of $K_{3,3}$.

Note. As the title of this section suggests, we are interested in the "crossings" of edges in a drawing of a graph. We need a definition that allows us to unambiguously count the number of edge crossings for a graph. We start with the following definitions.

Definition. A *crossing* in a drawing of a graph is a point shared by two edges in the drawing (excluding the end vertices of the edges). A crossing shared by more than two edges is a *multiple crossing.* A *simple drawing* of a graph in the plan is a drawing in which

(a) any two distinct edges have at most one crossing,

(b) any two edges incident with the same vertex do not cross, and

(c) no three edges cross in a common point (that is, there are no multiple crossings).

Note. Figures 9.1.3 and 9.1.4 give two drawings of $K_{3,3}$. The drawing in Figure 9.1.3 is not simple, since it violates each of (a), (b), and (c). The drawing in Figure 9.1.4 is a simple drawing of $K_{3,3}$.

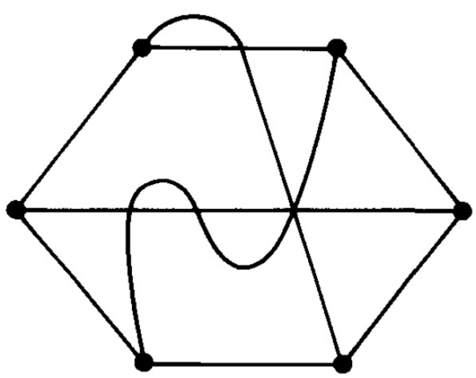

Figure 9.1.3

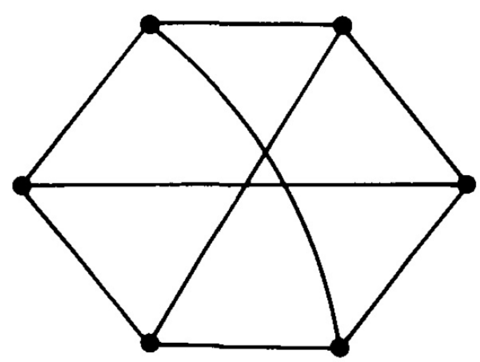

Figure 9.1.4

In the top row of Figure 9.1.5, three non-simple configurations in part of a drawing are given. Each is corrected in the second row of Figure 9.1.5.

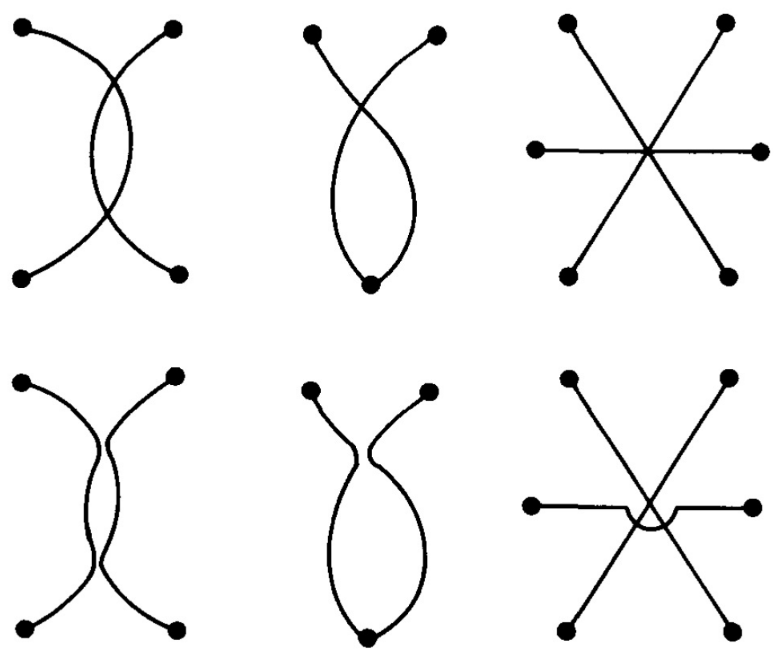

Figure 9.1.5

Definition. Let G be a graph. Among all possible drawings of G in the plane, consider one with the minimum number of crossings. This minimum number of crossings is if the *crossing number* of graph G, denoted $\text{cr}(G)$.

Note 9.1.B. A drawing of a graph G with $\text{cr}(G)$ is necessarily a simple drawing. The crossing number of a planar graph is 0 and the crossing number of a nonplanar graph is at least one. We know that K_5 and $K_{3,3}$ are nonplanar (by Theorems 8.1.4 and 8.1.6) and there exist drawings of both with only one crossing (consider Figures 9.1.1 and 9.1.2, for example) so that $\text{cr}(K_5) = 1$ and $\text{cr}(K_{3,3}) = 1$. Similar arguments can be used (along with Kuratowski's Theorem, Theorem 9.1.2) to find crossing numbers in some cases.

Problem 9.1.A. Find the crossing number of the following graph:

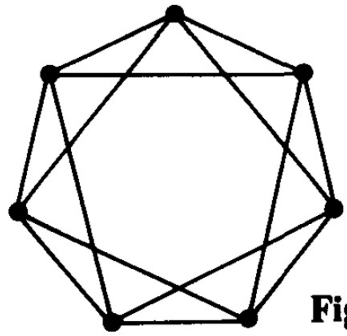

Figure 9.1.6

Solution. Trial and error will reveal that the graph can be drawn with one crossing (see Figure 9.1.7). We know by Kuratowski's Theorem (Theorem 9.1.2) that if the graph is not planar (and so has a crossing number of at least one) then it contains either a subdivision of K_5 or a subdivision of $K_{3,3}$. In Figure 9.1.8, we see that the graph contains a subdivision of $K_{3,3}$ (one partite set has white vertices and the other partite set consists of three of the four black vertices; the other black vertex results from subdividing an edge of $K_{3,3}$). Since the graph is nonplanar, then the crossing number is at least one, and since we have a simple drawing of the graph with exactly one crossing, then the crossing number is exactly one. □

Note. We know that K_4 is planar and so has a simple drawing with no crossings (see Note 9.1.B). We next prove that the number of crossing in a simple drawing of K_4 is limited.

Theorem 9.1.3 Every simple drawing of K_4 in the plane has either zero or one crossing.

Note. We can always put an upper bound on the crossing number of a graph by presenting a drawing (preferably a simple drawing; see Note 9.1.B) and counting the crossings. To show equality, we need an argument that no drawing can have fewer crossings then those demonstrated. In what follows, we mostly consider specific graphs which are "small" in some sense.

Theorem 9.1.4 The crossing number of K_6 is $\mathrm{cr}(K_6) = 3$.

Note. For complete graphs in general, the crossing number was addressed in Richard K. Guy's "A Combinatorial Problem," *Nabla* (*Bulletin of the Malayan Mathematical Society*), **7**, 68–72 (1960). He gave an upper bound on $\mathrm{cr}(K_n)$ of:

$$\mathrm{cr}(K_n) \leq \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Thomas L. Saaty in "The Minimum Number of Intersections in Complete Graphs," *Proceedings of the National Academy of Sciences*, **52**(3), 688–690 (1964) independently established this formula and proved that it reduces to equality for $n \leq 10$. Shengjun Pan and Bruce Richter in "The crossing number of K_{11} is 100," *Journal of Graph Theory*, **56**(2), 128–134 (2007) showed that equality holds for $n = 11$ and $n = 12$. That appears to be the status of the problem today (spring 2023). You can view Pan and Richter's paper on the Wiley Online Library webpage for the *Journal of Graph Theory* (you'll need your ETSU username and password to access the paper). The crossing numbers for $n = 5, 6, \ldots, 12$ are 1, 3, 9, 18, 36, 60, 100, 150, respectively. This historical information is based on the Wikipedia webpage on Crossing Number (accessed 12/31/2022).

Note. Kazimierz Zarankiewicz in "On a Problem of P. Turán Concerning Graphs," *Fundamenta Mathematicae*, **41**, 137–145 (1954) gave an upper bound on the crossing number of the complete bipartite graph $K_{m,n}$. He showed

$$\mathrm{cr}(K_{m,n}) \le \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

You can view Zarankiewicz's paper on the the Polish Digital Mathematics Library website. According to Hartsfield and Ringel, equality holds for $m \le 6$ and n arbitrary (see their page 186). We now prove this bound.

Theorem 9.1.5. Zarankiewicz's Theorem.
The crossing number of $K_{m,n}$ satisfies the inequality

$$\mathrm{cr}(K_{m,n}) \le \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Note. Finally, we prove that Zarankiewicz's Theorem reduces to equality when $m = 3$ and $m = 4$. It is conjectured to reduce to equality for all m and n.

Theorem 9.1.6. The crossing number of $K_{3,n}$ is $\mathrm{cr}(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

Theorem 9.1.7. The crossing number of $K_{4,n}$ is $\mathrm{cr}(K_{4,n}) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

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