Section 1.2. Topological Spaces

Note. In this section we state some fundamental definitions and quickly review the topics of continuity, homeomorphisms, basis for a topology, first and second countable topological spaces, and coarser/finer topologies.

Definition 2.1. A *topological space* is a set X together with a collection of subsets of X called *open sets* such that:

(1) the intersection of two open sets is open,

(2) the union of any collection of open sets is open, and

(3) the empty set $\varnothing$ and whole space X is open.

Additionally, a subset $C \subset X$ is *closed* if its complement $X - C$ is open.

Note. In a metric space, the metric determines the open and closed sets and hence the topology. Just as we can use the metric to define limits, convergence, continuity, we can use the topology to define these ideas. Other ideas such as limit points, boundary points, and compactness are also determined by the topology. We start with the definition of continuity of a function; notice that it only involves open sets.

Definition 2.2. If X and Y are topological spaces and $f : X \to Y$ is a function, then f is *continuous* if $f^{-1}(U)$ is open for each open set $U \subset Y$. A *map* is a continuous function.

Note. Since the complement of an open set is a curved set and since inverse images satisfy $f^{-1}(Y - B) = X - f^{-1}(B)$, then we have that $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed for each closed set $F \subset Y$.

Definition 2.3. If X is a topological space and $x \in X$ then a set N is a *neighborhood* of x in X is there is an open set $U \subset N$ with $x \in U$.

Definition 2.4. If X is a topological space and $x \in X$ then a collection $\mathbf{B}_x$ of subsets of X containing x is a *neighborhood basis* at x in X if each neighborhood of x in X contains some element of $\mathbf{B}_x$ and each element of $\mathbf{B}_x$ is a neighborhood of x.

Definition 2.5. A function $f : X \to Y$ between topological spaces is *continuous at* x, where $x \in X$, if, given any neighborhood N of $f(x)$ in Y, there is a neighborhood M of x n X such that $f(X) \subset N$.

Proposition 2.6. A function $f : X \to Y$ between topological spaces is continuous if and only if it is continuous at each point $x \in X$.

Definition 2.7. A function $f : X \to Y$ between topological spaces is a *homeomorphism* if $f^{-1} : Y \to X$ exists (i.e., f is one to one and onto) and both f and f^{-1} are continuous. The notation $X \approx Y$ means that X is homeomorphic to Y.

Definition 2.8. If X is a topological space and $\mathbf{B}$ is a collection of subsets of X, then $\mathbf{B}$ is a *basis* for the topology of X if the open sets are precisely the unions of members of $\mathbf{B}$. (Notice that the members of $\mathbf{B}$ are therefore open.) A collection $\mathbf{S}$ of subsets of X is a *subbasis* for the topology of X if the set $\mathbf{B}$ of finite intersections of members of $\mathbf{S}$ is a basis.

Definition 2.9. A topological space is *first countable* if each point has a countable neighborhood basis.

Definition 2.10. A topological space is *second countable* if its topology has a countable basis.

Definition 2.11. A sequence $f_1, f_2, \ldots$ of functions from a topological space X to a metric space Y *converges uniformly* to function $f : X \to Y$ if, for each $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that for all $i > n$ we have $\text{dist})f_i(x), f(x)) < \varepsilon$ for all $x \in X$.

Theorem 2.12. If a sequence $f_1, f_2, \ldots$ of continuous functions from a topological space X to a metric space Y converges uniformly to a function $f : X \to Y$, then f is continuous.

Definition 2.13. A function $f : X \to Y$ between topological spaces is *open* if $f(U)$ is open in Y for all open $U \subset X$. If is *closed* if $f(C)$ is closed in Y for all closed $C \subset X$.

Definition 2.14. If X is a set and some condition is given on subsets of X which may or may not hold for any particular subset, then if there is a topology T whose open sets satisfy the condition, and such that, for any topology T' whose open sets satisfy the condition, then the T-open sets are also T'-open (i.e., $T \subset T'$), then T is the *smallest* (or *weakest* or *coarsest*) topology satisfying the condition. If instead, for any topology T' whose open sets satisfy the condition, any T'-open sets are also T-open, then T is the *largest* (or *strongest* or *finest*) topology satisfying the condition.

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