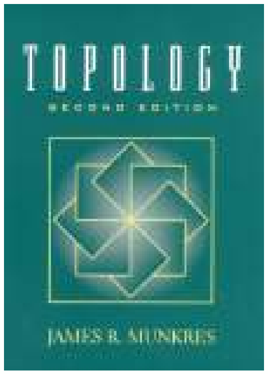
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 13. Basis for a Topology—Proofs of Theorems



Theorem 13.A

Theorem 13.A. Let $\mathcal{B}$ be a basis for a topology on X. Define

$$\mathcal{T} = \{U \subset X \mid x \in U \text{ implies } x \in B \subset U \text{ for some } B \in \mathcal{B}\},$$

the "topology" generated be $\mathcal{B}$. Then $\mathcal{T}$ is in fact a topology on X.

Proof. We consider the definition of "topology."

(1) $\varnothing \in \mathcal{T}$ vacuously. Now $X \in \mathcal{T}$ since each $x \in X$ satisfies $x \in B \subset X$ for some $B \in \mathcal{B}$ by the definition of topology generated by $\mathcal{B}$.

(2) Let $\{U_\alpha\}_{\alpha \in J}$ be an arbitrary collection of elements of $\mathcal{T}$. Let $U = \cup_{\alpha \in J} U_\alpha$. For $x \in U$ we have $x \in U_\alpha$ for some $\alpha \in J$. Since $U_\alpha \in \mathcal{T}$, then by the definition of "topology generated by $\mathcal{B}$," $x \in B \subset U_\alpha$ for some $B \in \mathcal{B}$. So $x \in B \subset U$ and hence by definition U is open.

Theorem 13.A (continued)

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the "topology" generated be $\mathcal{B}$. Then $\mathcal{T}$ is in fact a topology on X.

Proof (continued).

(3) Let $U_1, U_2 \in \mathcal{T}$. For $x \in U_1 \cap U_2$, by the definition of "topology generated by $\mathcal{B}$," there is $B_1 \subset U_1$ and $B_2 \subset U_2$ with $B_1, B_2 \in \mathcal{B}$ and $x \in B_1$, $x \in B_2$. By part (2) of the definition of "basis for a topology," there is $B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Hence $U_1 \cap U_2 \in \mathcal{T}$. Next, by mathematical induction, any finite collection $\{U_1, U_2, \ldots, U_n\} \subset \mathcal{T}$ satisfies $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$.

So $\mathcal{T}$ satisfies the definition of topology and $\mathcal{T}$ is a topology on X. □

Lemma 13.1

Lemma 13.1. Let X be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. As stated in Theorem 13.A above, all elements of $\mathcal{B}$ are open and so in $\mathcal{T}$. Since $\mathcal{T}$ is a topology, then by part (2) of the definition of "topology," any union of elements of $\mathcal{B}$ are in $\mathcal{T}$. So $\mathcal{T}$ contains all unions of elements of $\mathcal{B}$.

Next, suppose $U \in \mathcal{T}$. For each $x \in U$ choose $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ (which can be done by the definition of "topology $\mathcal{T}$ generated by $\mathcal{B}$"). Then $U = \cup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. Since U is an arbitrary element of $\mathcal{T}$, then all elements of $\mathcal{T}$ are unions of elements of $\mathcal{B}$ and the result follows. □

Lemma 13.2

Lemma 13.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open subset $U \subset X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

Proof. First we show that $\mathcal{C}$ is a basis. For the first part of the definition of basis, for $x \in X$ (since X itself is an open set) then (by hypothesis) there is $X \in \mathcal{C}$ such that $x \in C \subset X$. For the second part of the definition of basis, let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open then so is $C_1 \cap C_2$ by part (3) of the definition of topology. Then by hypothesis there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. So both parts of the definition of basis are satisfied and $\mathcal{C}$ is a basis for a topology on X.

Lemma 13.2 (continued)

Lemma 13.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open subset $U \subset X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

Proof (continued). Let $\mathcal{T}'$ be the topology on X generated by $\mathcal{C}$ (we now show that $\mathcal{T} = \mathcal{T}'$). First, if $U \in \mathcal{T}$ and $x \in U$, then since $\mathcal{C}$ is a basis for topology $\mathcal{T}$, there is $C \in \mathcal{C}$ such that $x \in C \subset U$. So, by the definition of "topology generated by" $\mathcal{C}$, U is in $\mathcal{T}'$ and hence $\mathcal{T} \subset \mathcal{T}'$. Conversely, if W belongs to $\mathcal{T}'$ then W is a union of elements of $\mathcal{C}$ by Lemma 13.1. Now each element of $\mathcal{C}$ is an element of $\mathcal{T}$ (see the comment at the end of the definition of "topology generated by") and a union of open sets is open (definition of topology, part (2)) so W belongs to $\mathcal{T}$. That is, $\mathcal{T}' \subset \mathcal{T}$. Therefore, $\mathcal{T} = \mathcal{T}'$. □

Lemma 13.3

Lemma 13.3. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}$ such that $x \in B' \subset B$.

Proof. (2)⇒(1) Given $U \in \mathcal{T}$, let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in C \subset U$. By hypothesis (2), there is $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$. Then $x \in B' \subset U$ and by the definition of "topology generated by" $\mathcal{B}'$, we have $U \in \mathcal{T}'$. So $\mathcal{T} \subset \mathcal{T}'$ and (1) follows.

(1)⇒(2) Let $x \in X$ and $B \in \mathcal{B}$ where $x \in B$. Since $\mathcal{B}$ generates $\mathcal{T}$, then $B \in \mathcal{T}$. By hypothesis (1), $\mathcal{T} \subset \mathcal{T}'$ and so $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is (by definition) $B' \in \mathcal{B}''$ such that $x \in B' \subset B$ and (2) follows. □

Lemma 13.4

Lemma 13.4. The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are each strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$, and $\mathbb{R}_K$, respectively. Given a basis element (a, b) for $\mathcal{T}$ and $x \in (a, b)$, the basis element $[x, b) \in \mathcal{T}'$ contains x and satisfies $[x, b) \subset (a, b)$. On the other hand, given basis element $[x, b)$ for $\mathcal{T}'$, there is no open interval (a, b) containing x which is a subset of $[x, d)$. Therefore $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ by Lemma 13.2.

Given a basis element (a, b) for $\mathcal{T}$ and $x \in (a, b)$, the same basis element $(a, b) \in \mathcal{T}'$ contains x and satisfies $(a, b) \subset (a, b)$. On the other hand, the basis element $B = (-1, 0) \setminus K$ for $\mathcal{T}''$ contains the point 0, but there is no open interval (a, b) containing 0 which is a subset of B. Therefore $\mathcal{T}''$ is strictly finer than $\mathcal{T}$ by lemma 13.2.

In Exercise 13.6 you will show that topologies $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable. □

Theorem 13.B

Theorem 13.B. Let $\mathcal{S}$ be a subbasis for a topology on X. Define $\mathcal{T}$ to be all unions of finite intersections of elements of $\mathcal{S}$. Then $\mathcal{T}$ is a topology on X.

Proof. Let $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{S}$:

$$\mathcal{B} = \{S_1 \cap S_s \cap \cdots \cap S_n \mid n \in \mathbb{N}; S_1, S_2, \ldots, S_n \in \mathcal{S}\}.$$

Let $x \in X$. Then $x \in S$ for some $S \in \mathcal{S}$ by the definition of subbasis, and so $x \in S$ where $S \in \mathcal{B}$. S part (1) of the definition of "$\mathcal{B}$ is a basis" is satisfied. Now let $B_1, B_2 \in \mathcal{B}$. Then $B_1 = S_2 \cap S_2 \cap \cdots \cap S_m$ and $B_2 = S_1' \cap S_2' \cap \cdots \cap S_n' \in \mathcal{B}$ and $B_2 \subset B_1 \cap B_2$ so that part (2) of the definition of "$\mathcal{B}$ is a basis" is satisfied and so $\mathcal{B}$ is a basis for a topology on X. The topology for which $\mathcal{B}$ is a basis is, by Lemma 13.1, the topology consisting of all unions of elements of $\mathcal{B}$. This is precisely the collection of sets in $\mathcal{T}$. So $\mathcal{T}$ is a topology on X. □