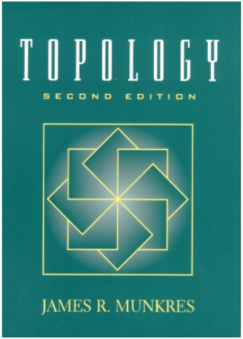
Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions

Section 13. Basis for a Topology—Proofs of Theorems



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Theorem 13.A

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$$\mathcal{T} = \{U \subset X \mid x \in U \text{ implies } x \in B \subset U \text{ for some } B \in \mathcal{B}\},$$

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(1) $\varnothing \in \mathcal{T}$ vacuously. Now $X \in \mathcal{T}$ since each $x \in X$ satisfies $x \in B \subset X$ for some $B \in \mathcal{B}$ by the definition of topology generated by $\mathcal{B}$.

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(2) Let $\{U_\alpha\}_{\alpha \in J}$ be an arbitrary collection of elements of $\mathcal{T}$. Let $U = \cup_{\alpha \in J} U_\alpha$. For $x \in U$ we have $x \in U_\alpha$ for some $\alpha \in J$. Since $U_\alpha \in \mathcal{T}$, then by the definition of "topology generated by $\mathcal{B}$," $x \in B \subset U_\alpha$ for some $B \in \mathcal{B}$. So $x \in B \subset U$ and hence by definition U is open.

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(3) Let $U_1, U_2 \in \mathcal{T}$. For $x \in U_1 \cap U_2$, by the definition of "topology generated by $\mathcal{B}$," there is $B_1 \subset U_1$ and $B_2 \subset U_2$ with $B_1, B_2 \in \mathcal{B}$ and $x \in B_1$, $x \in B_2$. By part (2) of the definition of "basis for a topology," there is $B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Hence $U_1 \cap U_2 \in \mathcal{T}$. Next, by mathematical induction, any finite collection $\{U_1, U_2, \ldots, U_n\} \subset \mathcal{T}$ satisfies $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$.

So $\mathcal{T}$ satisfies the definition of topology and $\mathcal{T}$ is a topology on X. □

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Lemma 13.1

Lemma 13.1. Let X be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. As stated in Theorem 13.A above, all elements of $\mathcal{B}$ are open and so in $\mathcal{T}$.

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Next, suppose $U \in \mathcal{T}$. For each $x \in U$ choose $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ (which can be done by the definition of "topology $\mathcal{T}$ generated by $\mathcal{B}$").

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Lemma 13.2

Lemma 13.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open subset $U \subset X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

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Proof (continued). Let $\mathcal{T}'$ be the topology on X generated by $\mathcal{C}$ (we now show that $\mathcal{T} = \mathcal{T}'$). First, if $U \in \mathcal{T}$ and $x \in U$, then since $\mathcal{C}$ is a basis for topology $\mathcal{T}$, there is $C \in \mathcal{C}$ such that $x \in C \subset U$. So, by the definition of "topology generated by" $\mathcal{C}$, U is in $\mathcal{T}'$ and hence $\mathcal{T} \subset \mathcal{T}'$.

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Lemma 13.3

Lemma 13.3. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}$ such that $x \in B' \subset B$.

Proof. (2)⇒(1) Given $U \in \mathcal{T}$, let $x \in U$.

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Proof. (2)$\Rightarrow$(1) Given $U \in \mathcal{T}$, let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in C \subset U$. By hypothesis (2), there is $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$.

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(1)$\Rightarrow$(2) Let $x \in X$ and $B \in \mathcal{B}$ where $x \in B$. Since $\mathcal{B}$ generates $\mathcal{T}$, then $B \in \mathcal{T}$. By hypothesis (1), $\mathcal{T} \subset \mathcal{T}'$ and so $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is (by definition) $B' \in \mathcal{B}''$ such that $x \in B' \subset B$ and (2) follows. □

Lemma 13.4

Lemma 13.4. The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are each strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$, and $\mathbb{R}_K$, respectively.

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Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$, and $\mathbb{R}_K$, respectively. Given a basis element (a, b) for $\mathcal{T}$ and $x \in (a, b)$, the basis element $[x, b) \in \mathcal{T}'$ contains x and satisfies $[x, b) \subset (a, b)$.

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Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$, and $\mathbb{R}_K$, respectively. Given a basis element (a, b) for $\mathcal{T}$ and $x \in (a, b)$, the basis element $[x, b) \in \mathcal{T}'$ contains x and satisfies $[x, b) \subset (a, b)$. On the other hand, given basis element $[x, b)$ for $\mathcal{T}'$, there is no open interval (a, b) containing x which is a subset of $[x, d)$. Therefore $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ by Lemma 13.2.

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Theorem 13.B

Theorem 13.B. Let $\mathcal{S}$ be a subbasis for a topology on X. Define $\mathcal{T}$ to be all unions of finite intersections of elements of $\mathcal{S}$. Then $\mathcal{T}$ is a topology on X.

Proof. Let $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{S}$:

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Let $x \in X$. Then $x \in S$ for some $S \in \mathcal{S}$ by the definition of subbasis, and so $x \in S$ where $S \in \mathcal{B}$. S part (1) of the definition of "$\mathcal{B}$ is a basis" is satisfied.

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