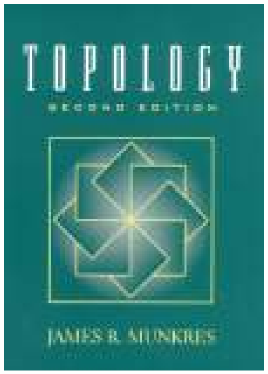
# Introduction to Topology

**Chapter 2. Topological Spaces and Continuous Functions**
Section 14. The Order Topology—Proofs of Theorems








## Theorem 14.A

**Theorem 14.A.** Let $X$ be a set with a simple order relation and let $\mathcal{B}$ consist of all open intervals $(a, b)$, all intervals $[a_0, b)$, and all intervals $(a, b_0]$, where $a_0$ is the least element of $X$ and $b_0$ is the greatest element of $X$ (if such exist). Then $\mathcal{B}$ is a basis for a topology on $X$.

**Proof.** We confirm that the definition of basis of a topology is satisfied. If $a_0$ is the least element of $X$ then $a_0 \in [a_0, b)$ for all $b \neq a_0$. If $b_0$ is the greatest element of $X$ then $b_0 \in (a, b_0]$ for all $a \neq b_0$. If $x \in X$ is neither the least nor greatest element of $X$ then $x \in (a, b)$ for some $a, b \in X$. So part (1) of the definition is satisfied.
For part (2) of the definition, let $B_1, B_2 \in \mathcal{B}$. Then $B_3 = B_2 \cap B_2 \in \mathcal{B}$ and $B_3 \subset B_1 \cap B_2$. So part (2) is satisfied and $\mathcal{B}$ is in fact a basis for a topology on $X$. □





## Theorem 14.B

**Theorem 14.B.** Let $X$ be a set with a simple order relation. The open rays form a subbasis for the order topology $\mathcal{T}$ on $X$.
**Proof.** Let $\mathcal{S}$ be the set of all open rays. As observed above, the open rays are in fact open sets in the order topology, so $\mathcal{S} \subset \mathcal{T}$ and the topology generated by $\mathcal{S}$ is a subset of $\mathcal{T}$ as well (Lemma 31.1). For the first part of the definition of subbasis, notice that $a < b$ implies that $X = (-\infty, b) \cup (a, \infty)$. For the second part of the definition of subbasis it is sufficient to show that every open interval $(a, b)$ is the intersection of a finite number of open rays. Of course $(a, b) = (-\infty, b) \cap (a, +\infty)$. (Notice that if $a_0$ is the least and $b_0$ is the greatest element of $X$ then $[a_0, +\infty) = (-\infty, b)$ and $(a, b_0] = (a, +\infty)$ are in fact open rays.) By Lemma 13.1, the open sets of the order topology consist of all unions of elements of $\mathcal{B} = \{(a, b) \mid a < b\}$, so the unions of all finite intersections of open rays in $\mathcal{S}$ include all open sets in the order topology. That is, the topology generated by $\mathcal{S}$ contains $\mathcal{T}$ and so the topology generated by $\mathcal{S}$ equals the order topology $\mathcal{T}$. □